Using a Bennett linkage as a connector between other Bennett loops*

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Abstract: The remarkable Bennett linkage has been used almost from the time of its announcement in 1903 as a fecund source of other kinematic loops and networks. Various techniques have been applied to yield an impressive number of products. Put forward here is a procedure by which the whole loop functions as an articulation between members of two other chains, resulting in a complex spatial network of a type having possible value as a deployable structure. An algebraic analysis also provides the determination of displacement–closure equations for two different forms of six-bar loop incorporated in the assemblage, one of them new to the literature.

Keywords: linkage kinematics, spatial linkages, deployable structures, overconstrained linkages, displacement–closure equations, mobile networks

1 BACKGROUND AND INTRODUCTION

First reported [**1**] a century ago and examined in great detail [**2**] a decade later, Bennett's four-revolute kinematic loop is the most celebrated of all spatial linkages, its geometrical and algebraic properties still being investigated today. Quite apart from its fascinating individual character, this overconstrained, mobile chain has been recognized from the beginning as a building-block for the construction of more complex forms. Bennett demonstrated [**2**] the mounting of multiloop networks based on his 'skew isogram mechanism' in analogy with planar ones developable from antiparallelogrammatic loops, as well as corresponding spherical constructions. With perhaps deeper insight, its implications shortly to be mentioned, he showed [**3**] that planar and spherical four-bars could be combined to produce hybrid six-bars by removing one joint of each after aligning them and fastening together newly adjacent links.

Inspired by Bennett's work, but anticipated [**4**] by Myard [**5**, **6**], Goldberg [**7**] introduced novel five- and six-bar linkages by adding and removing Bennett loops positioned so that common links and associated joints could be taken away without affecting mobility. Waldron [**8**] returned to Bennett's approach in synthesizing several new six-bar linkages, including one derived from a pair of Bennett chains. He also showed [**9**] that a whole loop could be substituted for a single articulation within another chain, a result of some significance to considerations in this paper. Wohlhart's concept of linkage 'isomers' [**10**] brought about by replacing one dyad of a Bennett linkage with the other, and the related 'Goldberg bridge', [**11**] expanded further the capability of the skew isogram for generating new chains. Variations on the foregoing methods are to be found in references [**12–14**].

Except for Bennett's deliberations on multiloop chains, the aforementioned works were all concerned with single-loop solutions. In contrast to the 'crossed' networks of reference [**2**], some preliminary attention [**15–19**] was accorded to 'mesh'

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assemblages formed from Bennett loops. The topic has been given fresh impetus recently from the field of deployable structures. These constructs bear a relationship to tiling arrangements and examples are provided by Chen [**20**] and Chen and You [**21**]. The present paper departs from what might be regarded as the mainstream development of the subject. Here, in a manner similar to Waldron's [**9**] technique, a set of Bennett loops is employed as a quartet of virtual joints between two other loops, resulting in a complex mobile network. Coincidental with the construction is the incorporation of two different types of six-bar loop. Because these linkages are mobile entities, they must be governed by appropriate sets of displacement–closure equations. A demonstration of the satisfaction of that requirement is undertaken successfully, and one of the forms is seen to be entirely new to the literature of kinematics.

2 KINEMATIC TOOLS

Bennett's skew four-bar, depicted skeletally in Fig. 1, comprises four links functioning as common perpendiculars between adjacent pairs of the four revolutes that connect them. The lengths of opposing members are equal, as are the non-consecutive angles of skew between succeeding joint axes. The two nominally independent link lengths and skew angles are related by the condition

$$a \sin\beta = b \sin\alpha$$

where

$$\beta \neq \alpha$$

Misinterpretations of the character of the loop, common in the literature, can be largely avoided by employing the right-hand screw rule stringently in defining angular displacements and restricting all angles of skew to lie in the range $(0, \pi)$. If the disposition of the linkage is such that it can be viewed approximately as a planar parallelogram, then [**22**] the axis orientations should be represented as displayed in Fig. 2, where it can be seen that one pair is directed oppositely to the other. That is the situation pertaining in this paper.

In what follows, sine, cosine, and tangent are abbreviated as s, c, and t, respectively. Because the loop has one degree of mobility, there can be only three independent displacement–closure equations governing the four variable joint angles θ_i. Two of them can be expressed by

$$\theta_1 + \theta_3 = 2\pi = \theta_2 + \theta_4 \quad (1), (2)$$

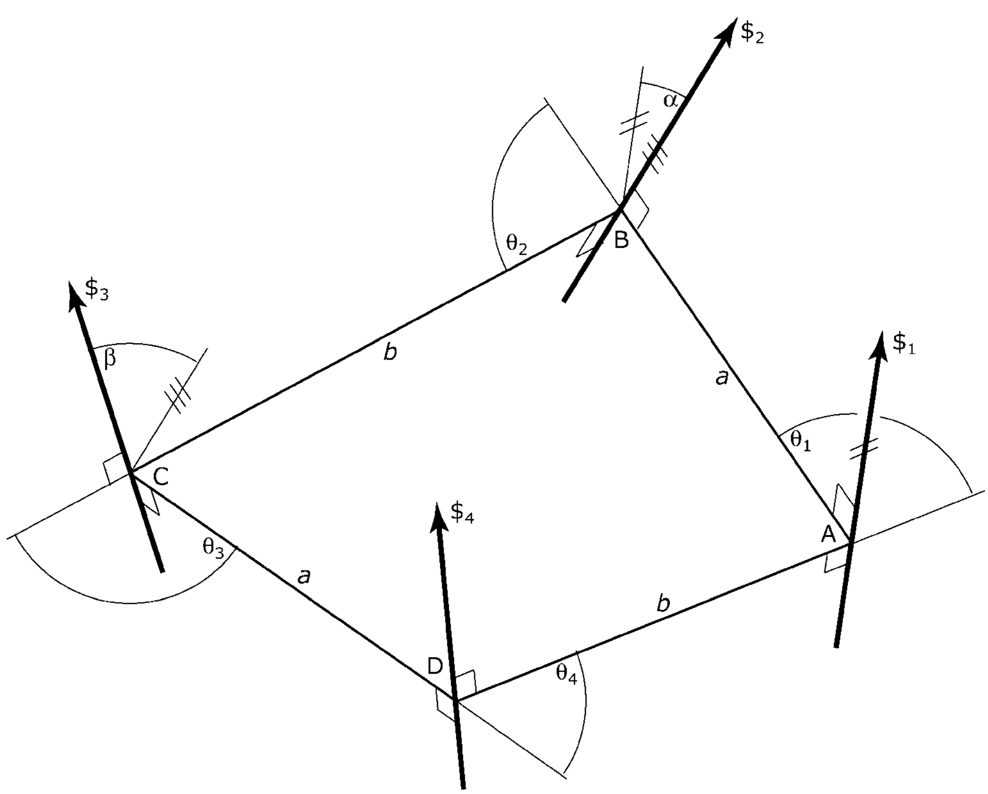

Fig. 1 Bennett linkage in schematic outline

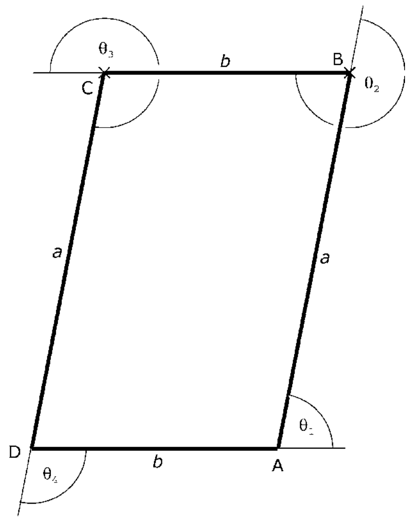

Fig. 2 Suitable representation of the Bennett loop when in uncrossed, quasi-planar manifestation

The third is most often given in the form

$$\mathrm{t}\frac{\theta_1}{2}\mathrm{t}\frac{\theta_2}{2} = \frac{\mathrm{s}[(\beta+\alpha)/2]}{\mathrm{s}[(\beta-\alpha)/2]} \tag{3}$$

but there are several alternative equations [**23**] available for relating adjacent joint angles, and it suits present purposes to supply an apposite selection here, as follows

$$\mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\beta - \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\alpha\mathrm{s}\beta - \mathrm{c}\theta_1\mathrm{s}\alpha\mathrm{c}\beta - \mathrm{c}\theta_2\mathrm{s}\alpha\mathrm{c}\beta - \mathrm{c}\alpha\mathrm{s}\beta = 0 \tag{4}$$

$$\mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\beta - \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{c}\alpha\mathrm{s}\beta + \mathrm{c}\theta_1\mathrm{s}\alpha + \mathrm{c}\theta_2\mathrm{s}\alpha + \mathrm{s}\beta = 0 \tag{5}$$

$$\mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\beta + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{c}\alpha\mathrm{s}\beta + \mathrm{s}\theta_1\mathrm{s}\alpha + \mathrm{s}\theta_2\mathrm{s}\alpha\mathrm{c}\beta = 0 \tag{6}$$

$$\mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\beta + \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\alpha\mathrm{s}\beta + \mathrm{s}\theta_1\mathrm{s}\alpha\mathrm{c}\beta + \mathrm{s}\theta_2\mathrm{s}\alpha = 0 \tag{7}$$

When fresh Bennett loops are used to modify an existing mobile assemblage after the style of Bennett, Goldberg, Waldron, or Wohlhart, as cited earlier, it is generally clear from physical considerations that internal motion is retained. If, however, interconnections are made between loops so that a new network is set up, then it is usually necessary to establish mobility by argument. Both circumstances obtain here. In regard to the latter, it transpires that six-bar chains are included as linking mechanisms, and so their mobility is essential to that of the whole assembly. Although one of the two six-bar types is readily handled in the analysis, the other calls for special treatment requiring the adhibition of the relevant displacement–closure relationships. These equations can be formulated in various ways. The set of 12 provided in Appendix 2 to this paper comprises nine equations of orientation, only three of which may be independent, and three of location. There is considerable flexibility available in their application. The variables and fixed parameters that appear in the relationships are best understood graphically, as portrayed in Fig. 3.

3 BUILDING THE NETWORK

The generic colligation laid out below is fabricated from 'similar' Bennett linkages. That is, the ratio of a pair of adjacent link lengths is always b/a (although the critical factor is actually [**22**] the quotient $\mathrm{t}(\beta/2)/\mathrm{t}(\alpha/2)$). The illustrative diagrams will be based upon Fig. 2 and angles of skew will be marked initially as positive, zero, or negative. This approach makes visualization of the network easier, but, in due course, skew angles and axis directions must be modified to conform with the conventions stated in section 2.

The procedure begins as shown in Fig. 4(a) where two Bennett loops, AJSP and CNRK, have been articulated to the original isogram ABCD at, respectively, J, P and N, K. For the present it may be taken that all joint axes are directed approximately out of the same plane. In order to make the new loops similar to the original one, the axes through J, B, K and

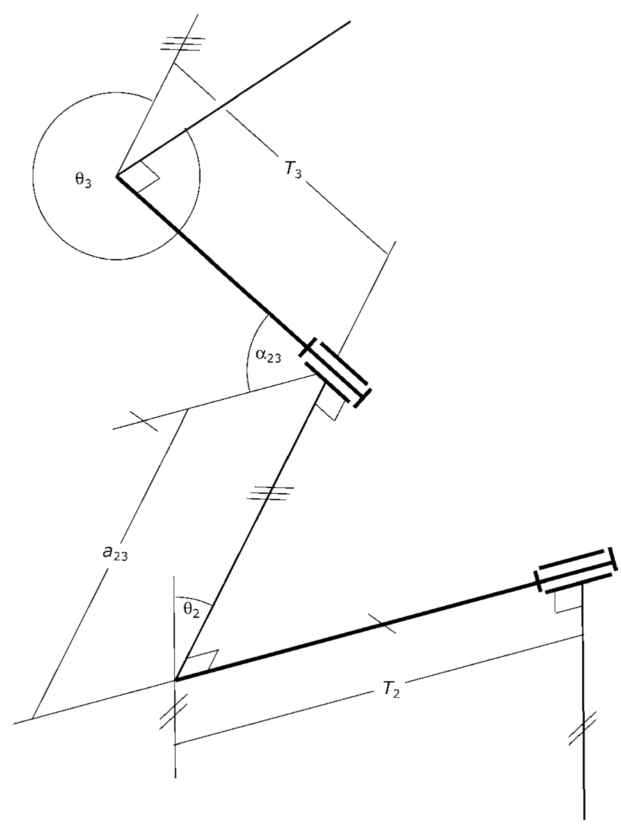

Fig. 3 Typical variables and parameters of a linkage containing only rotary joints

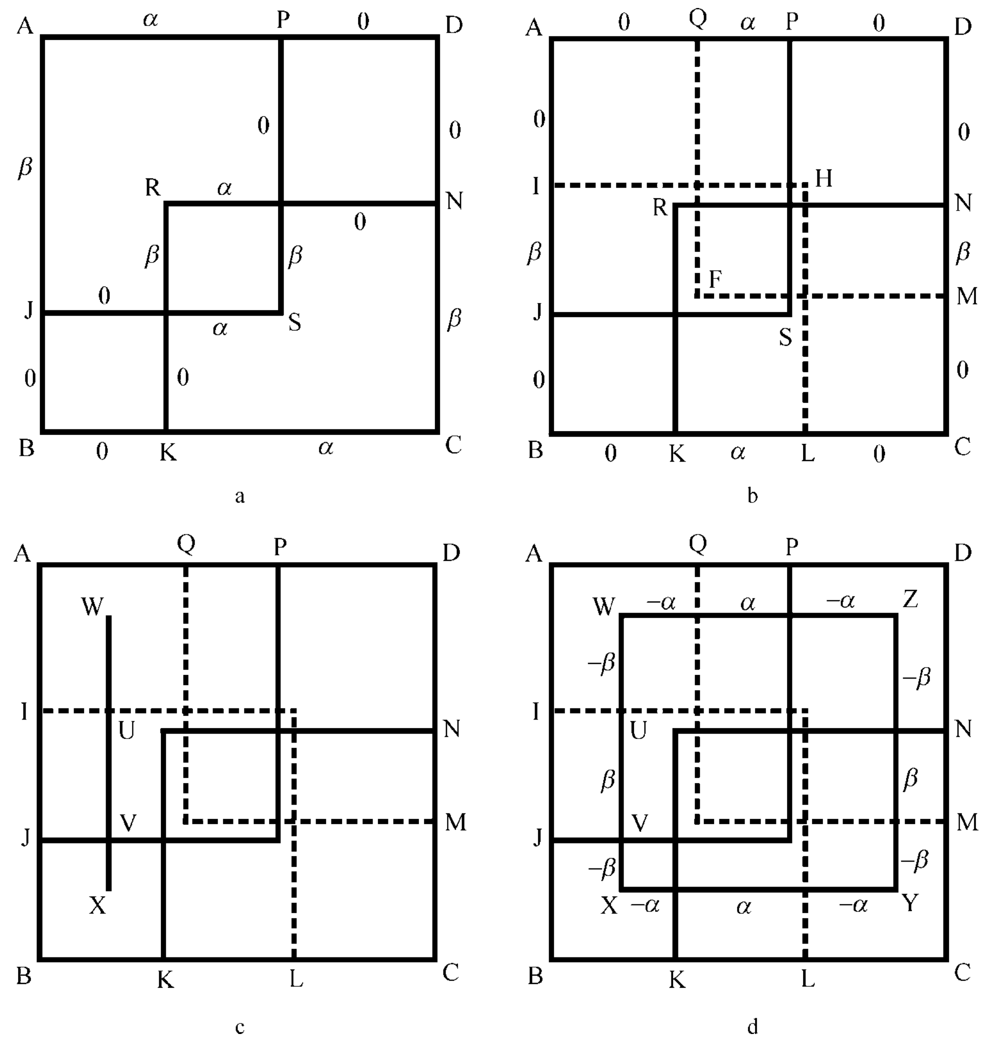

Fig. 4 Interconnection of a given Bennett linkage ABCD and a synthesized chain WXYZ: (a) insertion of two loops; (b) inclusion of a complementary pair; (c) an interconnecting member; (d) formation of the synthesized loop

those through N, D, P are parallel sets. Consequent angles of skew are indicated on the diagram. Link lengths for AJSP are chosen as k_1a, k_1b and those for CNRK as k_2a, k_2b. It is clear that these insertions have no effect on the motion capability of ABCD. There can now be introduced a complementary pair of chains, as marked by broken lines in Fig. 4(b). Again, loops BLHI and DQFM are Bennett linkages similar to ABCD, their link lengths proportional to those of ABCD with ratios k_3 and k_4, respectively. The mobility of ABCD remains unchanged. Then, the lengths of segments IJ, KL, MN, and PQ are

$$a_{IJ} = (k_1 + k_3 - 1)b \qquad a_{KL} = (k_2 + k_3 - 1)a$$
$$a_{MN} = (k_2 + k_4 - 1)b \qquad a_{PQ} = (k_1 + k_4 - 1)a$$

Next, bar WX is interposed (Fig. 4(c)) by connecting it to the existing assemblage at U and V with pin joints. To ensure retention of mobility, loop IJVU is to be a Bennett linkage similar to ABCD. Hence

$$a_{VU} = a_{IJ} = (k_1 + k_3 - 1)b$$

$$a_{UI} = a_{JV} = (k_1 + k_3 - 1)a$$

while the corresponding twists are β and α. The same can be done to the remaining sides of the construction (Fig. 4(d)). Each time, in order to retain mobility, the new loop is to be a Bennett linkage similar to ABCD. Finally, pin connections are made at W, X, Y, Z and, to satisfy the imposition of similarity once more, the twist marked alongside each segment in Fig. 4(d) is adopted. Loop WXYZ is then expected to be a Bennett linkage similar to ABCD with

$$a_{YZ} = a_{WX} = (3 - k_1 - k_2 - k_3 - k_4)b$$
$$a_{ZW} = a_{XY} = (3 - k_1 - k_2 - k_3 - k_4)a$$

and the corresponding twists $-\beta$ and $-\alpha$.

The network illustrated in Fig. 4(d) is a complex one. If the central part of the assembly is removed, it is evident that two prospective Bennett linkages, ABCD and WXYZ, are connected by four smaller ones (Fig. 5). All six loops are similar. Two working models of the network have been constructed. Their dimensions and the products are shown in Figs 6–9 More details of the constructions and the background to the procedure are available in references [**20**, **24**].

4 MOBILITY CONSIDERATIONS

Individually, each of the four-bars pictured in Fig. 5 is mobile, but there is no reason *a priori* for the whole colligation to be so. It might not have even been possible to construct it, being a very highly overconstrained assemblage. If, however, all of the subchains satisfy mobility criteria as well, then

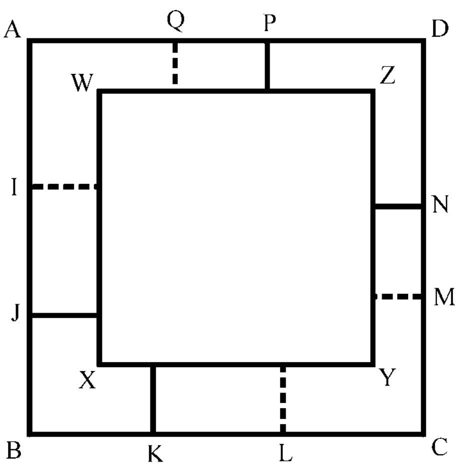

Fig. 5 Two ‘similar’ Bennett linkages connected by four others

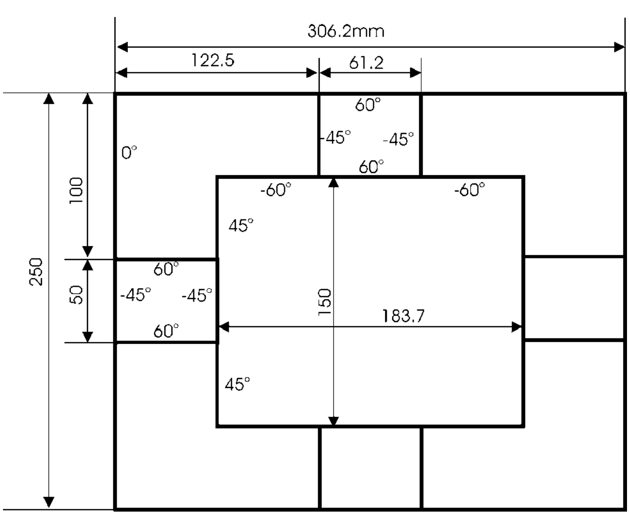

Fig. 6 Network model 1 – the design

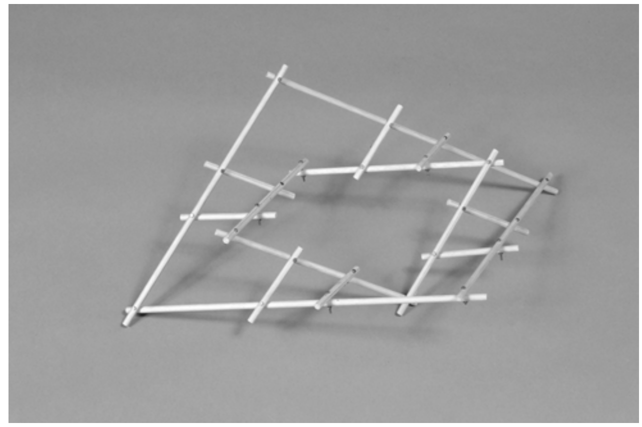

Fig. 7 Network model 1 – the realization

overall motion capability is assured. Attention is now restricted to that portion shown in Fig. 10 of the second network model. It is clear that this three-loop arrangement is representative of all such constituents. The reader’s attention is drawn to certain modifications from the original figure on the diagram. Negative angles of skew have been converted to positive ones with concomitant reversals of relevant joint axes. Then, as a consequence of equations (1) and (2) and the introduction of joint angles ϕ, ψ, all other joint angles but one are determinate, as indicated. The relationship between ϕ and ψ is expressible by any of equations (3)–(7). Finally, it is noted that

$$\frac{|OY|}{|YG|} = \frac{a}{b} = \frac{|KC|}{|CN|}$$

Figure 10 includes two distinct types of six-bar which are examined separately in what follows.

4.1 Known six-bar

Each of loops YOLCMG and YTKCNE can be recognized as a special case of the plano-Bennett hybrid [**8**], which is known to be mobile. There is no need to say more on this account, but, because it will be of assistance in the following subsection, some space is taken at this juncture to analyse, for example, the first chain named. It is detailed in Fig. 11 with the customary notation while remaining consistent with that of Fig. 10. The literature does not provide displacement–closure equations for the general plano-Bennett linkage as a set of five independent relationships, and so one is supplied below for the present case. The set is similar for loop YTKCNE, some changes in sign being necessary.

The angles of skew are already marked on the figure; in particular, it can be observed that the axes through L, C, M are parallel, the two contained

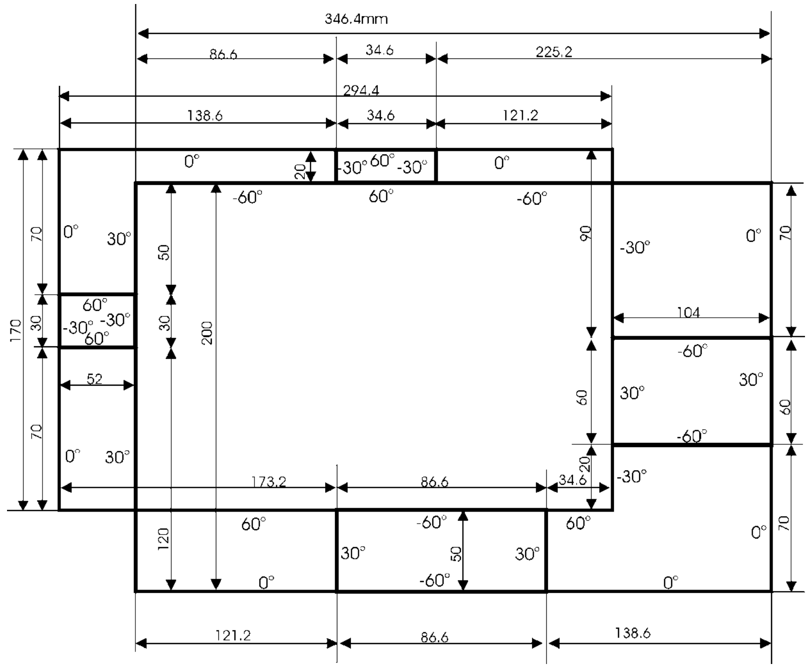

Fig. 8 Network model 2 – the design

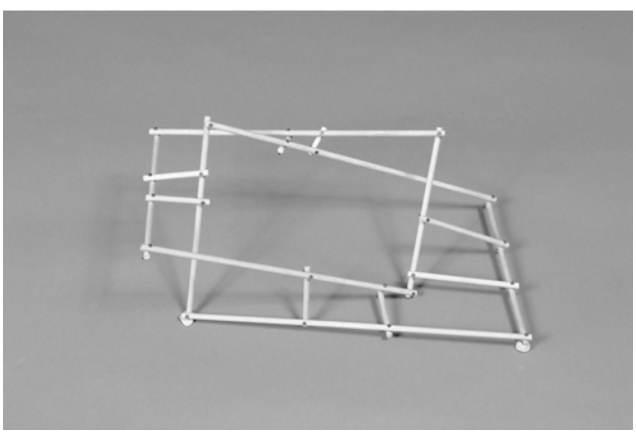

Fig. 9 Network model 2 – the realization

members and three axes comprising the planar component of the six-bar. Links GY, YO, and their three associated articulations make up the Bennett component. Conditions on link lengths can be stated immediately as

$$a_{45} = a_{61} + a_{23} \qquad a_{34} = a_{56} + a_{12}$$

$$a_{61}\mathrm{s}\alpha = a_{12}\mathrm{s}\beta$$

Owing to the mode of construction of the network, all offsets are necessarily zero, but a slight generalization can be accommodated whereby

$$T_6 = T_1 = T_2 = 0 = T_3 + T_4 + T_5$$

Directly from the figure, three of the five closure equations are

$$\theta_6 + \theta_2 = 2\pi \tag{8}$$

$$\theta_3 = \pi - \theta_4 = \theta_5 \qquad (9), (10)$$

Substitution of the dimensional constraints given earlier into equation (29) with indices advanced by 1 yields a fourth relationship, namely

$$\mathrm{s}\theta_6\mathrm{c}\theta_1\mathrm{s}\alpha - \mathrm{c}\theta_6\mathrm{s}\theta_1\mathrm{s}\alpha\mathrm{c}\beta - \mathrm{s}\theta_1\mathrm{c}\alpha\mathrm{s}\beta + \mathrm{s}\theta_6\mathrm{s}\beta = 0 \tag{11}$$

Considering next the Bennett loop MNEG, application of equation (7) leads to

$$\mathrm{c}\psi\mathrm{s}\phi\mathrm{s}\alpha + \mathrm{s}\psi\mathrm{c}\phi\mathrm{c}\beta\mathrm{s}\alpha - \mathrm{s}\psi\mathrm{s}\beta\mathrm{c}\alpha + \mathrm{s}\phi\mathrm{s}\beta = 0$$

which is equivalent to

$$\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{s}\alpha - \mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\beta\mathrm{s}\alpha - \mathrm{s}\theta_5\mathrm{s}\beta\mathrm{c}\alpha - \mathrm{s}\theta_6\mathrm{s}\beta = 0$$

Comparison of this result with equation (11) reveals that

$$\begin{aligned} \theta_1 &= \pi + \theta_5 \\ &= 2\pi - \psi \end{aligned} \tag{12}$$

The mobility of the other type of six-bar can now be more easily investigated.

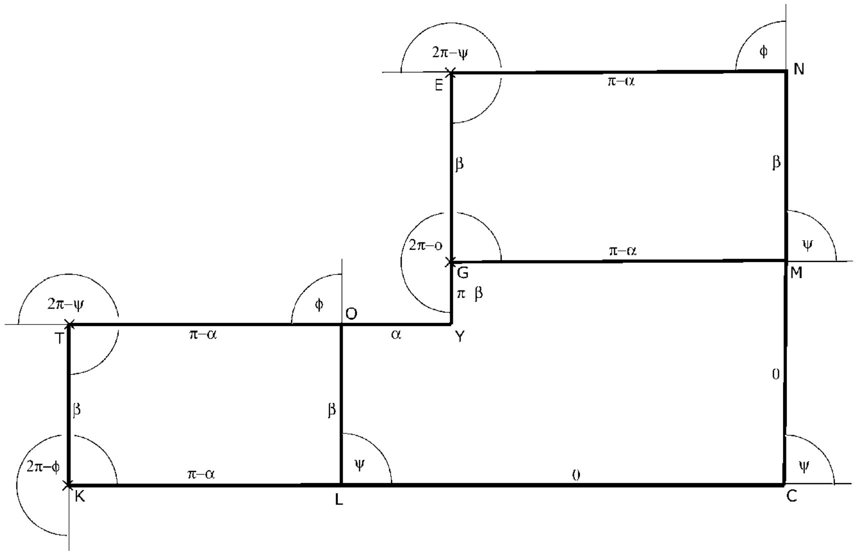

Fig. 10 Three-loop extract from the second network example with modified parameters

4.2 New six-bar

Chains YOLCNE and YTKCMG are of a kind that cannot be found in a recent listing [**25**] of all established six-revolute linkages. Their leading property is the existence of two pairs of parallel adjacent joint axes. The first loop is taken for scrutiny; apart from some changes in sign, the second yields like findings. The six-bar is represented in Fig. 12. Constraints on link lengths are seen to be

$$a_{45} = a_{61} + a_{23} \qquad a_{34} = a_{56} + a_{12}$$

$$a_{61}\mathrm{s}\alpha = a_{34}\mathrm{s}\beta$$

All offsets are again zero, but here a greater generalization can be accepted, namely

$$T_2 + T_5 = T_1 - T_6 = T_3 + T_4 = 0$$

It is unlikely, nevertheless, that the most general form of this family of six-bars has been thereby uncovered.

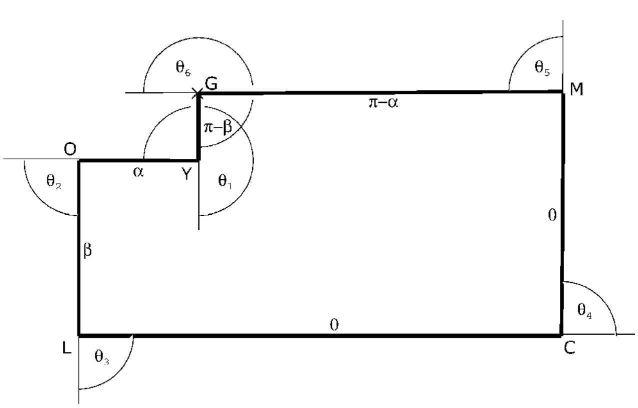

Fig. 11 Included plano-Bennett chain with standard notation

Four of the five requisite independent closure equations can be stated immediately as

$$\theta_6 = \theta_1 = \pi + \theta_3 = 2\pi - \theta_4 \qquad (13)\text{ to }(15)$$

$$\theta_2 + \theta_5 = \pi \qquad (16)$$

Substitution of the dimensional conditions into equation (29) yields

$$a_{12}\mathrm{s}\theta_2\mathrm{s}\beta - T_1\mathrm{c}\theta_2\mathrm{s}\alpha\mathrm{s}\beta + a_{56}\mathrm{s}\theta_5\mathrm{s}\beta + a_{61}(\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{s}\beta - \mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{s}\beta\mathrm{c}\alpha + \mathrm{s}\theta_6\mathrm{c}\beta\mathrm{s}\alpha) - T_6\mathrm{c}\theta_5\mathrm{s}\beta\mathrm{s}\alpha = 0$$

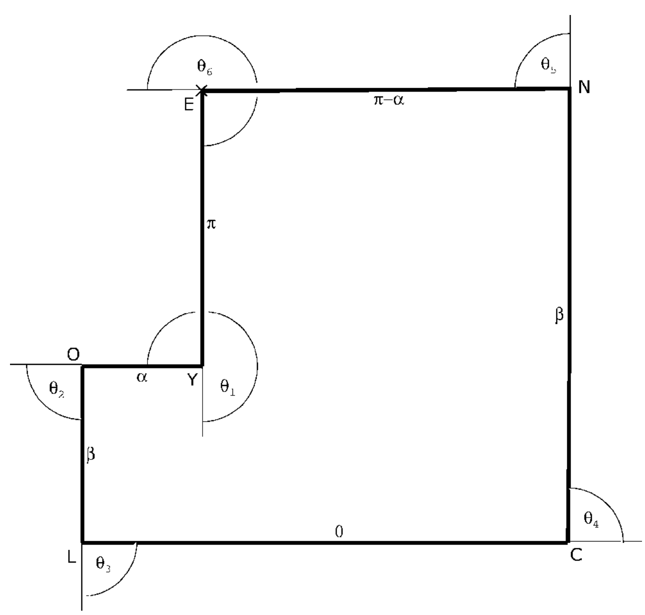

Fig. 12 Included six-bar of a type not previously described

which, after application of equation (16) and reuse of the dimensional constraints, leads to

$$\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{s}\beta - \mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{s}\beta\mathrm{c}\alpha + \mathrm{s}\theta_5\mathrm{s}\alpha + \mathrm{s}\theta_6\mathrm{c}\beta\mathrm{s}\alpha = 0 \quad (17)$$

This relationship is equivalent to

$$\mathrm{s}\phi\mathrm{c}\psi\mathrm{s}\beta + \mathrm{c}\phi\mathrm{s}\psi\mathrm{c}\alpha\mathrm{s}\beta + \mathrm{s}\phi\mathrm{s}\alpha - \mathrm{s}\psi\mathrm{s}\alpha\mathrm{c}\beta = 0$$

which, by virtue of equation (6), say, is also seen to be in accord with the integration of Bennett loop NEGM, as required. The set of closure relationships (13)–(17) is complete and is found, at length, to satisfy the other general equations (18)–(28).

5 SUMMARY REMARKS

This paper has been concerned, primarily, with the employment of whole Bennett linkages as a means of articulation between other loops to construct a mobile spatial network of some complexity. The number of free parameters available allows for considerable variation in the design, the potential of which in the field of deployable structures is yet to be exploited. Intrinsic to the network is the presence of 16 six-revolute linkages of two different kinds, all of which must be simultaneously mobile with the six Bennett chains. The analysis has revealed that one of the forms is an established linkage, but the other is described for the first time. Closure equations have been obtained for both types and the motion capability of the assemblage thereby proved.

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APPENDIX 1

Notation

a, a_{lm}, b	link lengths
k_n	link-length multipliers
T_i	offsets along joint axes
α, α_{lm}, β	angles of skew between pairs of adjacent joint axes
θ_i, ϕ, ψ	angular displacements about joint axes

APPENDIX 2

$$\begin{aligned}
&\mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6 - \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45} - \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{56}\\
&\quad - \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56} + \mathrm{s}\theta_4\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\\
&\qquad = \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_3 - \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{12} - \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{23}\\
&\qquad\quad - \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23} + \mathrm{s}\theta_1\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}
\end{aligned} \tag{18}$$

$$\begin{aligned}
&- \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{61} + \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{61}\\
&\quad - \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61} - \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\quad + \mathrm{s}\theta_4\mathrm{c}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61} + \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\quad + \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61} + \mathrm{s}\theta_4\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\qquad = \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_3 + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{12} - \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{23}\\
&\qquad\quad + \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23} - \mathrm{c}\theta_1\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{s}\alpha_{61} - \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{61}\\
&\quad + \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61} + \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\quad - \mathrm{s}\theta_4\mathrm{c}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61} + \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\quad + \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61} + \mathrm{s}\theta_4\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\qquad = \mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{s}\alpha_{12} + \mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23} + \mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6 + \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45} - \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{56}\\
&\quad + \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56} - \mathrm{c}\theta_4\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\\
&\qquad = -\mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{34} + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{34}\\
&\qquad\quad - \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34} - \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad\quad + \mathrm{s}\theta_1\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad\quad + \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34} + \mathrm{s}\theta_1\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&- \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61} + \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61}\\
&- \mathrm{c}\theta_4\mathrm{c}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61} + \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61}\\
&- \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61} - \mathrm{c}\theta_4\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\quad = -\mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{34} - \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{34}\\
&\qquad - \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34} + \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad - \mathrm{c}\theta_1\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad - \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34} - \mathrm{c}\theta_1\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}
\end{aligned} \tag{22}$$

$$\begin{aligned}
&+ \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61} - \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61}\\
&+ \mathrm{c}\theta_4\mathrm{c}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61} + \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61}\\
&- \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61} - \mathrm{c}\theta_4\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\quad = -\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{34} + \mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad + \mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} - \mathrm{c}\theta_2\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad + \mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\quad = \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{34} - \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{34}\\
&\qquad + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34} + \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad - \mathrm{s}\theta_1\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34} + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad + \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} + \mathrm{s}\theta_1\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&- \mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{61} + \mathrm{c}\theta_5\mathrm{c}\theta_6\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\quad + \mathrm{c}\theta_6\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61} - \mathrm{c}\theta_5\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\quad + \mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\quad = \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{34} + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{34}\\
&\qquad + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad - \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad + \mathrm{c}\theta_1\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34} + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad - \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} - \mathrm{c}\theta_1\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{61} - \mathrm{c}\theta_5\mathrm{c}\theta_6\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{s}\alpha_{61}\\
&\quad - \mathrm{c}\theta_6\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{s}\alpha_{61} - \mathrm{c}\theta_5\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\quad + \mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56}\mathrm{c}\alpha_{61}\\
&\qquad = \mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{34} - \mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad\quad - \mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34} - \mathrm{c}\theta_2\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad\quad + \mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}
\end{aligned} \tag{26}$$

$$\begin{aligned}
&a_{12}(\mathrm{c}\theta_2\mathrm{c}\theta_3 - \mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{23}) + T_1(\mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\\
&\quad + \mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23} + \mathrm{s}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23})\\
&\quad + a_{23}\mathrm{c}\theta_3 + T_2\mathrm{s}\theta_3\mathrm{s}\alpha_{23} + a_{34} + a_{45}\mathrm{c}\theta_4\\
&\quad + a_{56}(\mathrm{c}\theta_4\mathrm{c}\theta_5 - \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{c}\alpha_{45}) + T_5\mathrm{s}\theta_4\mathrm{s}\alpha_{45}\\
&\quad + a_{61}(\mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6 - \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45} - \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{56}\\
&\qquad - \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56} + \mathrm{s}\theta_4\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56})\\
&\quad + T_6(\mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{s}\alpha_{56} + \mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\\
&\quad + \mathrm{s}\theta_4\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}) = 0
\end{aligned} \tag{27}$$

$$\begin{aligned}
&a_{12}(-\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{c}\alpha_{34} - \mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34} + \mathrm{s}\theta_2\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34})\\
&\quad + T_1(-\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{34} + \mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\qquad + \mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} - \mathrm{c}\theta_2\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\qquad + \mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34})\\
&\quad - a_{23}\mathrm{s}\theta_3\mathrm{c}\alpha_{34} + T_2(\mathrm{c}\theta_3\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34} + \mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34})\\
&\quad + T_3\mathrm{s}\alpha_{34} + a_{45}\mathrm{s}\theta_4 + a_{56}(\mathrm{s}\theta_4\mathrm{c}\theta_5\\
&\quad + \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{c}\alpha_{45}) - T_5\mathrm{c}\theta_4\mathrm{s}\alpha_{45}\\
&\quad + a_{61}(\mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6 + \mathrm{c}\theta_4\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{c}\alpha_{45} - \mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{56}\\
&\qquad + \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56} - \mathrm{c}\theta_4\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56})\\
&\quad + T_6(\mathrm{s}\theta_4\mathrm{s}\theta_5\mathrm{s}\alpha_{56} - \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56}\\
&\quad - \mathrm{c}\theta_4\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56}) = 0
\end{aligned} \tag{28}$$

$$\begin{aligned}
&a_{12}(\mathrm{c}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{34} + \mathrm{s}\theta_2\mathrm{c}\theta_3\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34} + \mathrm{s}\theta_2\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34})\\
&\quad + T_1(\mathrm{s}\theta_2\mathrm{s}\theta_3\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{34} - \mathrm{c}\theta_2\mathrm{c}\theta_3\mathrm{s}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{s}\alpha_{34}\\
&\quad - \mathrm{c}\theta_3\mathrm{c}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34} - \mathrm{c}\theta_2\mathrm{s}\alpha_{12}\mathrm{s}\alpha_{23}\mathrm{c}\alpha_{34}\\
&\quad + \mathrm{c}\alpha_{12}\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34})\\
&\quad + a_{23}\mathrm{s}\theta_3\mathrm{s}\alpha_{34} + T_2(\mathrm{c}\alpha_{23}\mathrm{c}\alpha_{34} - \mathrm{c}\theta_3\mathrm{s}\alpha_{23}\mathrm{s}\alpha_{34})\\
&\quad + T_3\mathrm{c}\alpha_{34} + T_4 + a_{56}\mathrm{s}\theta_5\mathrm{s}\alpha_{45} + T_5\mathrm{c}\alpha_{45}\\
&\quad + a_{61}(\mathrm{s}\theta_5\mathrm{c}\theta_6\mathrm{s}\alpha_{45} + \mathrm{c}\theta_5\mathrm{s}\theta_6\mathrm{s}\alpha_{45}\mathrm{c}\alpha_{56} + \mathrm{s}\theta_6\mathrm{c}\alpha_{45}\mathrm{s}\alpha_{56})\\
&\quad + T_6(\mathrm{c}\alpha_{45}\mathrm{c}\alpha_{56} - \mathrm{c}\theta_5\mathrm{s}\alpha_{45}\mathrm{s}\alpha_{56}) = 0
\end{aligned} \tag{29}$$