Symmetric waterbomb origami

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The traditional waterbomb origami, produced from a pattern consisting of a series of vertices where six creases meet, is one of the most widely used origami patterns. From a rigid origami viewpoint, it generally has multiple degrees of freedom, but when the pattern is folded symmetrically, the mobility reduces to one. This paper presents a thorough kinematic investigation on symmetric folding of the waterbomb pattern. It has been found that the pattern can have two folding paths under certain circumstance. Moreover, the pattern can be used to fold thick panels. Not only do the additional constraints imposed to fold the thick panels lead to single degree of freedom folding, but the folding process is also kinematically equivalent to the origami of zero-thickness sheets. The findings pave the way for the pattern being readily used to fold deployable structures ranging from flat roofs to large solar panels.

1. Introduction

The waterbomb is a traditional origami (http://www.britishorigami.info/academic/lister/waterbomb.php). Commonly, two terms are related to it: waterbomb bases and waterbomb tessellations. There are two types of waterbomb bases: the eight-crease base and the six-crease base. The former is made from a square sheet of paper consisting of eight alternating mountain and valley creases around a central vertex (figure 1a). One of its typical tessellations is produced by four...
such bases tiling around a smaller square forming the square Resch pattern (figure 1b,c). The latter, consisting of two mountain and four valley creases shown in figure 1d, is more commonly known, and its tessellations range from a flat-foldable surface to a deformable tube known as the magic origami ball (figure 1e,f).

Both waterbomb origami structures were extensively investigated in the past. For instance, Hanna et al. and Bowen et al. established the bistable and dynamic model of the eight-crease waterbomb base [1,2]. Tachi et al. [3] worked on the rigidity of a six-crease origami tessellation with multiple degrees of freedom to achieve an adaptive freeform surface. On the application side, the first origami stent was made from the waterbomb tube aimed to achieve a large deployable ratio [4]. A worm robot [5] and a deformable wheel robot [6] were also proposed based on the magic origami ball.

In this paper, the focus is drawn on the six-crease waterbomb tessellation. Owing to its large deployable ratio between expanded and packaged states, it can be potentially used to fold large flat roofs and space solar panels. Although the waterbomb pattern is of multiple degrees of freedom, the symmetric folding is often preferred in most of research or artwork, which is done by constraining it with symmetric conditions and then controlling the motion to reach an ideal flat-foldable state. This is not easy in practice due to the fact that in rigid origami, the six-crease waterbomb base itself is a spherical 6R linkage with three degrees of freedom [7], thus the number of degrees of freedom for the pattern could increase significantly if the pattern consists of large number of such bases.

The waterbomb pattern is primarily created for zero-thickness sheets just like all of the origami patterns. Yet, in most of the practical engineering applications, the thickness of the material cannot simply be ignored. Various methods have been proposed to fold thick panel. In one instance, tapered surfaces are used to fold a thick panel using the Miura-ori of zero-thickness sheet [8], whereas in the other, offsets at the edge of the panels were introduced to implement folding of thick panels using the square-twist origami pattern [9]. More recent research suggested to replace folds with two parallel ones to accommodate the thickness of materials [10]. In all of the above methods, the fundamental kinematic model in which origami is treated as a series of interconnected spherical linkages remained. Different from the above methods, the authors of this paper have also proposed an approach in which the fold lines were only allowed to be placed on the top or the bottom of flat thick panels. As a result, the spherical linkage assembly for the origami of zero-thickness sheet is replaced by an assembly of spatial linkages. We have proved

Figure 1. Two waterbomb bases and their tessellations. (a) The eight-crease waterbomb base; (b) one of its tessellations forming the Resch pattern; (c) partially folded Resch pattern model; (d) the six-crease waterbomb base; (e) its tessellation in unfolded and folded states and (f) the tessellation can also be used to form a tube.
that not only are the assemblies of such panels foldable, but also they can be folded compactly under certain conditions [11].

In this paper, we provide a comprehensive kinematic analysis on foldability of the waterbomb tessellation made from the six-crease waterbomb bases of both a zero-thickness sheet and a panel of finite thickness. Kinematically, the folding of a zero-thickness sheet is modelled as spherical 6R linkages, whereas that of a thick panel is treated as an assembly of the Bricard linkages. The analysis has revealed a number of very interesting features associated with waterbomb origami, including the existence of two folding paths for general waterbomb origami of zero-thickness sheets when it is folded symmetrically. Moreover, because the Bricard linkages are overconstrained [12,13], the increase in the number of degrees of freedom occurring for the origami of zero-thickness sheet does not materialize for thick panels.

\[
\begin{array}{|c|c|}
\hline
\text{Table 1. Nomenclature.}^a \\
\hline
z_i & \text{coordinate axis of crease } i \text{ or revolute joint } i \\
x_i & \text{coordinate axis common normal from } z_{i-1} \text{ to } z_i \\
\alpha_{i(i+1)} & \text{angle of rotation from } z_i \text{ to } z_{i+1} \text{ about axis } x_{i+1}, \text{ also known as the twist of link } i(i + 1) \\
\alpha_i, \beta_i & \text{design angular parameters of the waterbomb origami pattern} \\
Q_{i(i+1)} & 3 \times 3 \text{ transformation matrix between the coordinate system of link } i(i - 1)i \text{ and that of link } i(i + 1) \text{ for spherical linkages} \\
\delta_i & \text{angle of rotation from } x_i \text{ to } x_{i+1} \text{ about axis } z_i \text{ in the vertex } D \text{ of the origami waterbomb pattern, also known as the revolute variable of joint } i \\
\varphi_i & \text{dihedral angle between link } (i - 1)i \text{ and link } i(i + 1) \text{ in the vertex } D \text{ of the waterbomb origami pattern} \\
\omega_i & \text{angle of rotation from } x_i \text{ to } x_{i+1} \text{ about axis } z_i \text{ in the vertex } W \text{ of the waterbomb origami pattern, also known as the revolute variable of joint } i \\
\phi_i & \text{dihedral angle between link } (i - 1)i \text{ and link } i(i + 1) \text{ in the vertex } W \text{ of the waterbomb origami pattern} \\
\delta_{ii+1}^W & \text{normal distance between } z_i \text{ and } z_{i+1}, \text{ also known as the length of link } i(i + 1) \text{ in the vertex } W \text{ of the thick-panel waterbomb pattern or the panel thickness} \\
\delta_{ii+1}^D & \text{normal distance between } z_i \text{ and } z_{i+1}, \text{ also known as the length of link } i(i + 1) \text{ in the vertex } D \text{ of the thick-panel waterbomb pattern or the panel thickness} \\
r_i^D & \text{normal distance between } x_i \text{ and } x_{i+1} \text{ also known as the offset of joint } i \text{ in the vertex } D \text{ of the thick-panel waterbomb pattern} \\
r_i^W & \text{normal distance between } x_i \text{ and } x_{i+1} \text{ also known as the offset of joint } i \text{ in the vertex } W \text{ of the thick-panel waterbomb pattern} \\
T_{i(i+1)} & 4 \times 4 \text{ transformation matrix between the coordinate system of link } i(i - 1)i \text{ and that of link } i(i + 1) \\
a & \text{thickness parameter for thick-panel waterbomb pattern, also the thickness of link } 23 \text{ in the vertex } W \text{ of the thick-panel waterbomb pattern} \\
\mu & \text{ratio between the thickness of link } 34 \text{ and link } 23 \text{ in the vertex } W \text{ of the thick-panel waterbomb pattern} \\
\delta_i' & \text{angle of rotation from } x_i \text{ to } x_{i+1} \text{ about axis } z_i \text{ in the vertex } D \text{ of the thick-panel waterbomb pattern, also known as the revolute variable of joint } i \\
\varphi_i' & \text{dihedral angle between link } (i - 1)i \text{ and link } i(i + 1) \text{ in the vertex } D \text{ of the thick-panel waterbomb pattern} \\
\omega_i' & \text{angle of rotation from } x_i \text{ to } x_{i+1} \text{ about axis } z_i \text{ in the vertex } W \text{ of the thick-panel waterbomb pattern, also known as the revolute variable of joint } i \\
\phi_i' & \text{dihedral angle between link } (i - 1)i \text{ and link } i(i + 1) \text{ in the vertex } W \text{ of the thick-panel waterbomb pattern} \\
\hline
\end{array}
\]

\(^a\text{The set-up of coordinates and kinematic parameters for both zero-thickness and thick-panel origami according to Denavit—Hartenberg’s (DH) notation is shown in figure 11 of appendix A.}\)
2. Symmetric rigid folding of the waterbomb pattern of zero-thickness sheet

Consider a pattern made by tessellating six-crease waterbomb bases (figure 2a). The pattern consists of only two types of vertices, D and W, enlarged in figure 2b,c. The rigid origami folding around each vertex can be modelled kinematically as a spherical 6R linkage in which the creases act as revolute joints and the sheets between creases are rigid links. In general, a spherical 6R linkage is of three degrees of freedom, but this number is reduced to one if only the symmetric folding is allowed. In such a way, vertex D is regarded as a spherical 6R linkage with the geometrical parameters \( \alpha_{12} = \alpha_{34} = \alpha_{45} = \alpha_{61} = \alpha, \alpha_{23} = \alpha_{56} = \pi - 2\alpha, \) where \( 0 < \alpha \leq \pi / 2. \) Imposing the line and plane symmetry conditions, i.e. \( \delta_1 = \delta_4 \) and \( \delta_2 = \delta_3 = \delta_5 = \delta_6, \) to the closure condition of the linkage (see appendix A), we can then write the closure equations as

\[
\tan \frac{\delta_1}{2} = -\cos \alpha \tan \frac{\delta_2}{2} \quad \text{and} \quad \delta_1 = \delta_4, \delta_2 = \delta_3 = \delta_5 = \delta_6. \tag{2.1}
\]

Similarly, applying the symmetry condition to vertex W, it becomes a plane-symmetric spherical 6R linkage with the geometric parameters \( \alpha_{12} = \alpha_{61} = \pi - \alpha - \beta, \alpha_{23} = \alpha_{56} = \beta, \alpha_{34} = \alpha_{45} = \alpha, \) where \( 0 < \beta \leq \pi / 2, \) and

\[
\omega_5 = \omega_3 \quad \text{and} \quad \omega_6 = \omega_2. \tag{2.2a}
\]

To ensure the compatibility of the entire pattern, the kinematic relationship between \( \omega_1 \) and \( \omega_3 \) of vertex W must be identical to that between \( \delta_1 \) and \( \delta_2 \) of vertex D. Replacing \( \delta_1 \) and \( \delta_2 \) in equation (2.1) with \( \omega_1 \) and \( \omega_3, \) respectively, yields

\[
\tan \frac{\omega_1}{2} = -\cos \alpha \tan \frac{\omega_3}{2}. \tag{2.2b}
\]

Now, considering the closure condition of the linkage at W, we obtain two sets of equations. The first set is

\[
\tan \frac{\omega_2}{2} = -\frac{\cos \alpha}{\cos(\alpha + \beta)} \tan \frac{\omega_3}{2} \tag{2.3a}
\]

and

\[
\omega_4 = \omega_1. \tag{2.3b}
\]

whereas the second one is

\[
\tan \frac{\omega_2}{2} = -\frac{2 \sin \alpha \tan(\omega_3/2)}{\sin(\beta - \alpha) \tan^2(\omega_3/2) + \sin(\alpha + \beta)}. \tag{2.4a}
\]

and

\[
\tan \frac{\omega_4}{2} = \frac{\tan(\omega_3/2)(-2 \cos \alpha \sin^2(\beta - \alpha) \tan^4(\omega_3/2) + 4(\sin \alpha \sin 2\beta \\
+ \cos \alpha (\sin \alpha + \beta) \sin(\beta - \alpha)) \tan^2(\omega_3/2) + \sin(\alpha + \beta)(7 \sin \beta - \sin(2\alpha + \beta))) \\
- \cos 2\beta \tan^2(\omega_3/2) - 2 \sin^2(\alpha + \beta)}{2 \sin(\beta - \alpha)(2 \sin(\alpha + \beta) + \sin(\beta - \alpha)) \tan^4(\omega_3/2) + 4(\cos^2(\alpha + \beta) \\
+ \cos \alpha (\sin \alpha + \beta) \sin(\beta - \alpha)) \tan^2(\omega_3/2) + \sin(\alpha + \beta)(7 \sin \beta - \sin(2\alpha + \beta)))}. \tag{2.4b}
\]

Together with equations (2.1) and (2.2), the entire set of closure equations of the waterbomb pattern has been obtained.
The kinematic variables, or rotations about each crease, can be replaced by the dihedral angles between adjacent sheets connected by the crease. The relationship between the kinematic variables and dihedral angels is \( \delta_1 = \pi - \phi_1, \delta_2 = \pi + \phi_2, \delta_3 = \pi + \phi_3, \delta_4 = \pi - \phi_4, \delta_5 = \pi + \phi_5, \delta_6 = \pi + \phi_6 \) for vertex D and \( \omega_1 = \pi - \phi_1, \omega_2 = \pi - \phi_2, \omega_3 = \pi + \phi_3, \omega_4 = \pi - \phi_4, \omega_5 = \pi + \phi_5, \omega_6 = \pi - \phi_6 \) for vertex W. Thus, the two sets of kinematic relationships of the waterbomb pattern presented by the dihedral angels become

\[
\tan \frac{\phi_1}{2} = \frac{1}{\cos \alpha} \tan \frac{\phi_3}{2}, \hspace{1cm} (2.5a)
\]
\[
\tan \frac{\phi_2}{2} = \frac{\cos(\alpha + \beta)}{\cos \alpha} \tan \frac{\phi_3}{2}, \hspace{1cm} (2.5b)
\]
\[
\phi_4 = \phi_1, \hspace{0.5cm} \phi_5 = \phi_3, \hspace{0.5cm} \phi_6 = \phi_2, \hspace{1cm} (2.5c)
\]
\[
\psi_2 = \phi_3, \hspace{1cm} (2.5d)
\]
\[
\tan \frac{\psi_1}{2} = \frac{1}{\cos \alpha} \tan \frac{\psi_2}{2}, \hspace{0.5cm} \psi_1 = \psi_4, \hspace{0.5cm} \psi_2 = \psi_3 = \psi_5 = \psi_6 \hspace{1cm} (2.5e)
\]

and

\[
\tan \frac{\phi_1}{2} = \frac{1}{\cos \alpha} \tan \frac{\phi_3}{2}, \hspace{1cm} (2.6a)
\]
\[
\tan \frac{\phi_2}{2} = \frac{\sin(\alpha + \beta) \tan^2(\phi_3/2) + \sin(\beta - \alpha)}{2 \sin \alpha \tan(\phi_3/2)}, \hspace{1cm} (2.6b)
\]
\[
\tan(\phi_3/2)(2\sin^2(\alpha + \beta) \tan^2(\phi_3/2) - 4(\cos^2(\alpha + \beta) - \cos 2\beta) \tan^2(\phi_3/2) - 2\sin(\beta - \alpha)(2\sin(\alpha + \beta) + \sin(\beta - \alpha))) \sin(\alpha + \beta)(7\sin \beta - 2\sin(2\alpha + \beta)) \tan^2(\phi_3/2) + 4(\sin \alpha \sin 2\beta + \cos \alpha \sin(\alpha + \beta) \sin(\beta - \alpha)) \tan^2(\phi_3/2) - 2\cos \alpha \sin^2(\beta - \alpha)
\]
\[
\phi_4 = \phi_1, \hspace{0.5cm} \phi_5 = \phi_3, \hspace{0.5cm} \phi_6 = \phi_2, \hspace{1cm} (2.6c)
\]
\[
\psi_2 = \phi_3, \hspace{1cm} (2.6d)
\]

and

\[
\tan \frac{\psi_1}{2} = \frac{1}{\cos \alpha} \tan \frac{\psi_2}{2}, \hspace{0.5cm} \psi_1 = \psi_4, \hspace{0.5cm} \psi_2 = \psi_3 = \psi_5 = \psi_6 \hspace{1cm} (2.6e)
\]
Figure 3. Kinematic behaviour of the waterbomb origami pattern with $\alpha = 2\pi/9$, $\beta = 2\pi/9$. Kinematic relationships of vertices (a) $W$ and (b) $D$ (c) two folding paths with configurations i–viii.

Considering a pattern with $\alpha = 2\pi/9$, $\beta = 2\pi/9$, and taking $\phi_1$ as an input, the variations of other dihedral angles at vertex $W$ with respect to $\phi_1$ are plotted in figure 3a. There are two paths with the same starting point ($\pi$, $\pi$) and ending point (0, 0): path I based on equations (2.5a–e) and path II on equations (2.6a–f). It indicates that vertex $W$ can be folded compactly along two different paths. Yet for vertex $D$, with $\phi_1 = \phi_4 = \phi_1$, there is only one path (figure 3b). Therefore, in general, the patterns with a large number of vertices $D$ and $W$ will fold in two different ways, from i, ii, iii, iv to v, or from i, viii, vii, vi to v, as demonstrated in figure 3c.
There are a few special cases of the waterbomb pattern which are most interesting. First, when $\alpha + \beta = \pi/2$, creases along $z_2$ and $z_6$ at vertex $W$ shown in figure 2c become collinear. As a result, they fold together like a single crease. Path I, given by equation (2.5), breaks down into two straight lines. A particular case with $\alpha = \beta = \pi/4$ is shown in figure 4. At the first folding stage, $\phi_2$ (and $\phi_6$) starts from $\pi$ and finishes at 0 from i, xi, x to ix, whereas $\phi_1$, $\phi_3$, $\phi_4$ and $\phi_5$ remain to be $\pi$, then $\phi_2$ (and $\phi_6$) is kept at constant 0 and $\phi_1$, $\phi_3$, $\phi_4$ and $\phi_5$ change from $\pi$ to 0 along ix, viii, vii, vi and v. Both reach the compactly folded configuration. At the latter stage, vertex $W$ behaves like

Figure 4. Two-stage motion of path I with $\alpha = \pi/4$, $\beta = \pi/4$. (a) Folding paths with configurations i–xi and (b) kinematic relationships of vertex $W$. 


Figure 5. Blockage of waterbomb origami pattern. (a) Kinematic curve between $\phi_4$ and $\phi_1$ of unit $W$ with $\alpha = 7\pi/36$, $\beta = \pi/4$; (b) folding manners with $\alpha = 7\pi/36$, $\beta = \pi/4$; (c) folding manners with $\alpha = \pi/6$, $\beta = \pi/3$, in which the framed configurations are with physical blockage.

Second, equations (2.5) or (2.6) could give negative dihedral angles, which indicates a blockage occurring during folding, because physically the dihedral angles cannot be less than zero. By analysing equation (2.5b), it can be found that for path I when $\alpha + \beta > \pi/2$, $\phi_2$ is always negative.
Figure 5. (Continued.)

except at points \((0, 0)\) and \((\pi, \pi)\). So a blockage is always there. From equation \((2.6c)\), it can be
found that on path II when \(\alpha \neq \beta\), a blockage will occur when

\[
\frac{1}{\cos \alpha} \sqrt{-2 \left( \frac{\sin \alpha \sin 2\beta}{\cos \alpha \sin(\alpha + \beta) \sin(\beta - \alpha)} \right) + \frac{4(\sin \alpha \sin 2\beta + \cos \alpha \sin(\alpha + \beta) \sin(\beta - \alpha))^2}{\sin(\alpha + \beta)(7 \sin \beta - \sin(2\alpha + \beta))}} \\
< \tan \phi_1/2 < \sqrt{\frac{\cos^2(\alpha + \beta) - \cos 2\beta + \sqrt{(\cos^2(\alpha + \beta) - \cos 2\beta)^2 + \sin(\beta - \alpha) \sin^2(\alpha + \beta)(2 \sin(\alpha + \beta) + \sin(\beta - \alpha))}}}{\cos \alpha \sin(\alpha + \beta)}} .
\]

For example, when \(\alpha = 7\pi/36\), \(\beta = \pi/4\), the kinematic curve between \(\phi_4\) and \(\phi_1\) is shown in
figure 5a, and the folding sequences are demonstrated in figure 5b. Along path I, the pattern
can be folded from a sheet at i to fully folded configuration at vii, whereas along path II, the
folding process terminates at iii. The framed configurations are physically impossible owing to
blockage, because these configurations correspond to cases where \(\phi_4\) becomes negative. Even if
the penetrations were allowed, folding along path II would end up in a fully folded configuration
at vi that differs from that at vii along path I.

The physical blockage can also occur when \(\alpha + \beta = \pi/2\) but \(\alpha \neq \beta\). Figure 5c shows a two-stage
motion on path I and a blockage on path II for a pattern with \(\alpha = \pi/6\) and \(\beta = \pi/3\). Based on the
above analysis, the behaviour of the waterbomb tessellation can be summarized as follows.
(a) When $\alpha + \beta < \pi/2$ and $\alpha = \beta$, there are two smooth folding paths with neither two-stage motion nor blockage.
(b) When $\alpha + \beta < \pi/2$ and $\alpha \neq \beta$, path II is blocked and path I is smooth.
(c) When $\alpha + \beta = \pi/2$ and $\alpha = \beta$, path I is in two-stage motion, whereas path II is smooth.
(d) When $\alpha + \beta = \pi/2$ and $\alpha \neq \beta$, both two-stage motion on path I and blockage on path II happen.
(e) When $\alpha + \beta > \pi/2$ and $\alpha = \beta$, only path II for vertex $W$ is smooth, but vertex $D$ is blocked. Thus, the whole pattern is blocked from compact folding.
(f) When $\alpha + \beta > \pi/2$ but $\alpha \neq \beta$, both paths are blocked.

Among them, only cases (a)–(c) can have one or two smooth folding paths.

### 3. Folding thick panels with the waterbomb pattern

The waterbomb tessellation can also be used to fold panels with non-zero thickness. This is done by mapping the same pattern onto a thick panel while placing the fold lines either on top or bottom surfaces of the panel. Now, at $D$ and $W$, there will still be six fold lines in places of creases, but these fold lines no longer converge to a vertex. In other words, dissimilar to the zero-thickness sheet, the distances between the adjacent fold lines are no longer zeros. In terms of the kinematic model, the spherical 6$R$ linkage is now replaced by spatial 6$R$ linkages. Among all possible spatial 6$R$ linkages, the plane-symmetric Bricard linkage [13,15] is the most suitable one [11]. Let us select two Bricard linkages for $D$ and $W$, respectively, figure 6$a,b$, with their link lengths being the panel thicknesses. As the linkages are overconstrained, the geometrical conditions of the linkage at $D$ are

$$a_{12}^D = a_{61}^D = a_{34}^D = a_{45}^D = (2 + \mu)a, \quad a_{23}^D = a_{56}^D = 0,$$

$$a_{12}^D = 2\pi - \alpha, \quad a_{61}^D = \alpha, \quad a_{23}^D = \pi - 2\alpha, \quad a_{56}^D = \pi + 2\alpha, \quad a_{34}^D = \alpha, \quad a_{45}^D = 2\pi - \alpha$$

and $R_i^D = 0(i = 1, 2, 3, 4, 5, 6)$; (3.1c)

and those at $W$ are

$$a_{12}^W = a_{61}^W = (1 + \mu)a, \quad a_{23}^W = a_{56}^W = a, \quad a_{34}^W = a_{45}^W = \mu a,$$

$$a_{12}^W = \pi - \alpha - \beta, \quad a_{61}^W = \pi + \alpha + \beta, \quad a_{23}^W = \beta, \quad a_{56}^W = 2\pi - \beta,$$

$$a_{34}^W = 2\pi - \alpha, \quad a_{45}^W = \alpha$$

and $R_i^W = 0(i = 1, 2, 3, 4, 5, 6)$, (3.2c)

Here, $\alpha$ and $\beta$ are the same as the sector angles of the origami pattern in figure 2$b,c$ and $a_{ij}^D$ and $a_{ij}^W$ are expressed using the DH notation [14], whereas $a$ is the thickness of link 23 and $\mu$ is the proportion between the thickness of link 34 and link 23 in the vertex $W$ of the thick-panel waterbomb pattern where $a \neq 0$ and $\mu \neq 0$. Applying the closure condition of the linkages leads to the following closure equations (see appendix A). For $D$, two sets of closure equations can be obtained, which are

$$\tan \frac{\delta_1'}{2} = -\frac{1}{\cos \alpha} \tan \frac{\delta_2'}{2}, \quad \delta_3' = \delta_2' + \pi, \quad \delta_4' = \delta_1', \quad \delta_5' = \delta_3', \quad \delta_6' = \delta_2'$$

and

$$\tan \frac{\delta_1'}{2} = \frac{2 \cos \alpha \tan(\delta_2'/2)}{\tan^2(\delta_2'/2) - \cos 2\alpha'}, \quad \delta_3' = \pi - \delta_2', \quad \delta_4' = -\delta_1', \quad \delta_5' = \delta_3', \quad \delta_6' = \delta_2'$$

respectively. The relationship between the kinematic variables and dihedral angels at $D$ is $\delta_1' = 2\pi - \phi_1, \delta_2' = \phi_2, \delta_3' = \pi + \phi_3, \delta_4' = 2\pi - \phi_4, \delta_5' = \pi + \phi_5, \delta_6' = \phi_6$. By conversion of the kinematic
variables to the dihedral angels, the two sets of closure equations can be respectively rewritten as

\[
\begin{align*}
\tan \frac{\phi_2'}{2} &= \frac{1}{\cos \alpha} \tan \frac{\phi_3'}{2}, \\
\phi_4 &= \phi_1, \quad \phi_2' = \phi_3 = \phi_5 = \phi_6 
\end{align*}
\] (3.5a, b)
and
\[
\tan \frac{\phi'_1}{2} = \frac{2 \cos \alpha \tan(\phi'_2/2)}{-\tan^2(\phi'_2/2) + \cos 2\alpha} \quad (3.6a)
\]
\[
\phi'_3 = -\phi'_2, \quad \phi'_4 = -\phi'_1, \quad \phi'_5 = \phi'_3, \quad \phi'_6 = \phi'_2. \quad (3.6b)
\]
Similarly, we also have two sets of closure equations at \(W\), which are
\[
\tan \frac{\omega'_1}{2} = -\frac{1}{\cos \alpha} \tan \frac{\omega'_3}{2}, \quad (3.7a)
\]
\[
\tan \frac{\omega'_2}{2} = \frac{\cos(\alpha + \beta)}{\tan(\omega'_3/2)}, \quad (3.7b)
\]
\[
\omega'_4 = \omega'_1, \quad \omega'_5 = \omega'_3, \quad \omega'_6 = \omega'_2. \quad (3.7c)
\]
and
\[
\tan \frac{\omega'_1}{2} = \frac{-\tan(\omega'_3/2)(\mu \sin^2(\alpha + \beta) \tan^2(\omega'_2/2) + (\mu + 1)(\mu \sin^2 \beta + \sin^2 \alpha))}{\sin(\alpha + \beta)(\mu^2 \sin \beta + \cos(\alpha + \beta) \sin \alpha) \tan^2(\omega'_3/2) + (\mu + 1)^2 \sin \alpha \sin \beta \cos \beta}, \quad (3.8a)
\]
\[
\tan \frac{\omega'_2}{2} = \frac{(\mu + 1) \sin \alpha/((\mu \sin(\alpha + \beta))}{\tan(\omega'_3/2)}, \quad (3.8b)
\]
\[
\tan \frac{\omega'_4}{2} = \frac{-\tan(\omega'_3/2)(4\mu^2 \sin \alpha \sin^2(\alpha + \beta) \tan^2(\omega'_2/2) - 4(\mu + 1) \sin \alpha((\mu + 1) \sin^2 \beta - \sin^2(\alpha + \beta)))}{(\cos(3\alpha + \beta) - 2(1 + \mu^2 \cos(\alpha + \beta) + (1 + 4\mu + 2\mu^2) \cos(\alpha - \beta)) \sin(\alpha + \beta)) \tan^2(\omega'_3/2) + 2(\mu + 1)^2 \sin^2 \alpha \sin 2\beta}, \quad (3.8c)
\]
\[
\omega'_5 = \omega'_3, \quad \omega'_6 = \omega'_2. \quad (3.8d)
\]
The above two sets of closure equations can be written in terms of dihedral angles. Noting that the relationship between the kinematic variables and dihedral angles at \(W\) is \(\omega'_1 = 2\pi - \phi'_1, \omega'_2 = \pi - \phi'_2, \omega'_3 = \phi'_3, \omega'_4 = 2\pi - \phi'_4, \omega'_5 = \phi'_5, \omega'_6 = \pi - \phi'_6\), the two sets of closure equations now become
\[
\tan \frac{\phi'_1}{2} = \frac{1}{\cos \alpha} \tan \frac{\phi'_3}{2}, \quad (3.9a)
\]
\[
\tan \frac{\phi'_2}{2} = \frac{\cos(\alpha + \beta)}{\cos \alpha} \tan \frac{\phi'_3}{2}, \quad (3.9b)
\]
\[
\phi'_4 = \phi'_1, \quad \phi'_5 = \phi'_3, \quad \phi'_6 = \phi'_2. \quad (3.9c)
\]
and
\[
\tan \frac{\phi'_1}{2} = \frac{\tan(\phi'_3/2)(\mu \sin^2(\alpha + \beta) \tan^2(\phi'_2/2) + (\mu + 1)(\mu \sin^2 \beta + \sin^2 \alpha))}{\sin(\alpha + \beta)(\mu^2 \sin \beta + \cos(\alpha + \beta) \sin \alpha) \tan^2(\phi'_3/2) + (\mu + 1)^2 \sin \alpha \sin \beta \cos \beta}, \quad (3.10a)
\]
\[
\tan \frac{\phi'_2}{2} = \frac{\mu \sin(\alpha + \beta)}{(\mu + 1) \sin \alpha} \tan \frac{\phi'_3}{2}, \quad (3.10b)
\]
\[
\tan \frac{\phi'_4}{2} = \frac{-\tan(\phi'_3/2)(4\mu \sin \alpha \sin^2(\alpha + \beta) \tan^2(\phi'_2/2) - 4(\mu + 1) \sin \alpha((\mu + 1) \sin^2 \beta - \sin^2(\alpha + \beta)))}{(\cos(3\alpha + \beta) - 2(1 + \mu^2 \cos(\alpha + \beta) + (1 + 4\mu + 2\mu^2) \cos(\alpha - \beta)) \sin(\alpha + \beta)) \tan^2(\phi'_3/2) + 2(\mu + 1)^2 \sin^2 \alpha \sin 2\beta}, \quad (3.10c)
\]
and \(\phi'_5 = \phi'_3, \quad \phi'_6 = \phi'_2. \quad (3.10d)\)

So far, two complete sets of closure equations have been obtained. It can be noted from all closure equations that the motions of the linkages retain the plane symmetry. Additional compatibility
conditions between the vertices $D$ and $W$ need to be added, which are
\[ \phi_1' = \varphi_1' \quad \text{and} \quad \phi_3' = \varphi_2'. \] (3.11)

We shall now discuss the respective motion paths provided by two sets of closure equations.

— The first set of closure equations, equation (3.5), at $D$ and the first set of closure equations, equation (3.9) at $W$.

Because equations (3.5a) and (3.9a) are identical, the compatibility between $D$ and $W$, equation (3.11), is satisfied automatically. Therefore, there is always a smooth folding path for the thick-panel origami for any $\mu \neq 0$, figure 7a–c, in which $\mu$ is randomly selected as 0.5. By comparing equations (3.5) and (3.9) for the thick panel with equations (2.5) for the zero-thickness sheet, we can conclude that the thick-panel origami and the path $I$ of the original waterbomb origami pattern are kinematically identical, as demonstrated by the folding sequence of the physical models in figure 7d. The motions of both structures are line and plane symmetric. Moreover, when $\alpha + \beta = \pi/2$, path $I$ becomes a two-stage motion, where $\phi_2'$ and $\phi_6'$ change from $\pi$ to 0, whereas $\phi_1', \phi_3', \phi_4'$, and $\phi_5'$ are kept to $\pi$, followed by the process that $\phi_1', \phi_3', \phi_4'$, and $\phi_5'$ move as a spatial 4R linkage. This linkage is actually a Bennett linkage. It eventually reaches the compact folding position. However, blockage could be occurred during the motion owing to the panel thickness, which could prevent the structure from being fully folded, see figure 8, in which $\mu$ is randomly selected as 0.7.

— The first set of closure equations, equation (3.5), at $D$ and the second set of closure equations, equation (3.10) at $W$.

Consider equations (3.5a) and (3.10a). Under the compatibility condition given by equation (3.11), there must be
\[ \mu = \frac{\cos(\alpha + \beta) \sin \alpha}{\sin \beta}. \] (3.12a)

Additionally, when $\alpha = \beta$, another solution exists, which is
\[ \mu = 1. \] (3.12b)

Under the first solution given in (3.12a), equation (3.10) effectively coincides with equation (3.9), and thus there is only one set of closure equations for $W$. Only one folding path exists as shown in figure 9 for the case where $\alpha = 7\pi/36$, $\beta = \pi/4$ and $\mu = 0.14$. Note that this path matches that shown in figure 7c despite that in the latter, $\mu$ is randomly selected as 0.5. The motion behaviour of the thick-panel waterbomb remains the same as the zero-thickness origami in path $I$, and thus we name it path $I$ for thick-panel origami. Moreover, when $\alpha + \beta = \pi/2$, $\mu = 0$ from equation (3.12a). So it will not be considered.

Under the second solution, $\mu = 1$, given by (3.12b), equations (3.9) and (3.10) are different. In other words, together with equation (3.5), there are two sets of closure equations for the thick-panel origami with $\mu = 1$ that result in two folding paths. The first, based on equations (3.5) and (3.9), has been discussed earlier. The second, based on equations (3.5) and (3.10), is actually identical to equation (2.6) of the zero-thickness sheet. This shows that the corresponding folding path is kinematically identical to the path $II$ of the waterbomb origami pattern of the zero-thickness sheet, and thus it is named as path $II$ of the thick-panel origami. One of such example is shown in figure 10.

In thick-panel origami, there is also blockage because of collision of panels during the folding process. Generally, along path $I$ of $W$, the blockage would appear when one of the dihedral angles becomes negative. The condition without blockage is $\phi_2' > 0$. Considering equation (3.9b) leads to $\alpha + \beta < \pi/2$, which is the same conclusion as the zero-thickness origami pattern summarized in last section. To avoid the interference at $D$ during the folding, $0 < \alpha \leq \pi/4$ must be satisfied.

— The second set of closure equations, equation (3.6), at $D$.

The other set of closure equations given by equation (3.6) at $D$ signify that in the thick-panel case, there exists a folding path that violates the line symmetry. However, this path is practically always blocked, because $\varphi_3'$ and $\varphi_2'$, $\varphi_4'$, and $\varphi_1'$ always have opposite signs as indicated by equation (3.6b).
Therefore, the behaviour of the general thick-panel waterbomb can be summarized as follows.

(a) For any \( \mu \neq 0 \), when \( \alpha + \beta < \pi / 2 \), there is only one smooth folding path: \textit{path I}.

(b) For any \( \mu \neq 0 \), when \( \alpha + \beta = \pi / 2 \), there is one two-stage folding path, \textit{path I}, with blockage.

**Figure 7.** Kinematic paths of thick panel waterbomb when \( \alpha = 7\pi / 36 \), \( \beta = \pi / 4 \), \( \mu = 0.5 \). Kinematic relationships at vertices (a) W and (b) D with \( \phi'_1 \) taken as input, where vertex W works as a plane symmetric Bricard linkage, whereas vertex D works as a line and plane symmetric Bricard linkage; (c) folding path with configurations i–v; (d) folding sequences of physical models of zero-thickness sheets and thick panels.
Figure 7. (Continued.)

Figure 8. Folding path of thick panel waterbomb pattern with $\alpha = \pi/6$, $\beta = \pi/3$, $\mu = 0.7$, in which the framed configurations are with physical blockage.
Figure 9. Folding path of thick-panel waterbomb pattern with $\alpha = 7\pi/36$, $\beta = \pi/4$ and $\mu = \cos(\alpha + \beta) \cdot \sin \alpha / \sin \beta = 0.14$.

(c) For any $\mu \neq 0$, when $\alpha + \beta > \pi/2$, there is one blocked folding path.
In particular,
(d) For $\mu = 1$, when $\alpha + \beta < \pi/2$, $\alpha = \beta$, there are two smooth folding paths, kinematically equivalent to paths I and II in the zero-thickness origami.
(e) For $\mu = 1$, when $\alpha = \beta = \pi/4$, path I is in two-stage motion and blocked, but path II can achieve smooth folding.

Here, paths I and II cannot be switched from one to another once the motions are underway. The choice of folding paths has to be made at the start and end configurations. The detailed comparison on the kinematic behaviour of the general waterbomb tessellation of zero-thickness sheets and thick panels for different design parameters is given in table 2 of appendix B.

4. Conclusions and discussion

In this paper, we have analysed the rigid origami of the waterbomb tessellation of both zero-thickness sheets and thick panels under the symmetric motion condition. By introducing the plane-symmetric Bricard linkages to replace the spherical 6R linkages in the origami pattern, the thick-panel waterbomb structure has been successfully formed. The rigorous enforcement of compatibility conditions ensures the mobility and flat-foldability of the thick-panel origami. We have also proven that the thick-panel origami and that of the zero-thickness sheet are kinematically equivalent.

Despite the fact that the thick-panel origami is born from an existing origami of zero-thickness sheet, it has a number of advantages over its parent. First, kinematically the thick-panel origami structure is a mobile assembly of overconstrained Bricard linkages with only one degree of freedom, and thus no additional constraints are required to keep its motion symmetrical. This
Figure 10. Folding sequence for patterns with $\alpha = \beta = \frac{2\pi}{9}$ and $\mu = 1$. (a) Two folding paths exist; physical models of zero-thickness sheet (top) and thick panel that fold (b) along paths I and (c) II, respectively.

could be a great benefit for real engineering applications as its control system could become much more simple and reliable. Second, in general, the origami of waterbomb tessellation for zero-thickness sheets has kinematic singularity when it is flat and fully compact. However, for thick-panel origami, the singularity only appears when a very specific thickness is chosen. A suitable selection of the thickness of the panels make the latter possible to achieve compact folding without bifurcations. The unique motion path is certainly much desirable for most practical applications.

The waterbomb tessellation for the thick panels enables the structure to be folded compactly. The compactness of the package depends on the thickness coefficient and the number of vertices
within the pattern. The pattern can be divided into strips formed by vertices $D$ in the horizontal direction. Consider a pattern consisting of $m$ strips, each with $n$ vertices $D$. In the completely packaged configuration, the dimension in the vertical direction will be $(m + 1)/2$ of the height of the larger triangles in vertex $D$ and the cross-section dimensions are the width of the larger triangles in vertex $D$ and the overall thickness as $2n(2 + 2\mu)a$, where $n$ is the number of vertices $D$ in the strip and $\mu \leq 1$. $\mu > 1$ is not recommended because it results in panels with considerable thickness and, in turn, the overall thickness of the package when the panels are packaged. So the ratio between the area of a fully expanded shape and that of completely folded is about $4n$. This indicates that the concept is very suitable to fold a structure in a long rectangular shape. On the other hand, to meet the geometrical conditions of the spatial linkages, each panel within the pattern could not be of the same thickness. As a result, the overall structure in the fully deployed configuration is flat but not absolutely even. However, for this waterbomb pattern, we have managed to make sure that one side of the expanded surface is completely flat, which enables the waterbomb origami pattern to be directly applicable to fold thick-panel structures such as solar panels and space mirrors.

Data accessibility. This work does not have any experimental data.

Authors’ contributions. Y.C. and Z.Y. initiated the project, worked on this topic and wrote the paper. H.F. conducted all the equation derivation under the supervision of Y.C. and J.M. R.P. constructed all the models. J.M. did all the verification between the analytical and modelling results. All authors gave final approval for publication.

Competing interests. We have no competing interests.

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Appendix A

According to the DH notation set-up in figure 11, the transformation matrix can be assembled as

$$
T_{(i+1)i} = \begin{bmatrix}
\cos \theta_i & -\cos \alpha_{(i+1)} \sin \theta_i & \sin \alpha_{(i+1)} \sin \theta_i & a_{(i+1)} \cos \theta_i \\
\sin \theta_i & \cos \alpha_{(i+1)} \cos \theta_i & -\sin \alpha_{(i+1)} \cos \theta_i & a_{(i+1)} \sin \theta_i \\
0 & \sin \alpha_{(i+1)} & \cos \alpha_{(i+1)} & R_i \\
0 & 0 & 0 & 1
\end{bmatrix},
$$

(A 1)

which transforms the expression in the $i$ + 1th coordinate system to the $i$th coordinate system.

The inverse transformation can be expressed as

$$
T_{i(i+1)} = \begin{bmatrix}
\cos \theta_i & \sin \theta_i & 0 & -a_{i(i+1)} \\
-\cos \alpha_{(i+1)} \sin \theta_i & \cos \alpha_{(i+1)} \cos \theta_i & \sin \alpha_{(i+1)} \cos \theta_i & R_i \sin \alpha_{(i+1)} \\
\sin \alpha_{(i+1)} \sin \theta_i & -\sin \alpha_{(i+1)} \cos \theta_i & \cos \alpha_{(i+1)} & -R_i \cos \alpha_{(i+1)} \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

(A 2)

For a single loop linkage consisting of six links, the closure equation is

$$
T_{21} \cdot T_{32} \cdot T_{43} = T_{61} \cdot T_{56} \cdot T_{45}.
$$

(A 3)

As for spherical linkages, the axes intersect at one point, which means the lengths of each links are zeros and thus equation (A 3) reduces to

$$
Q_{21} \cdot Q_{32} \cdot Q_{43} = Q_{61} \cdot Q_{56} \cdot Q_{45},
$$

(A 4)

where

$$
Q_{(i+1)i} = \begin{bmatrix}
\cos \theta_i & -\cos \alpha_{(i+1)} \sin \theta_i & \sin \alpha_{(i+1)} \sin \theta_i \\
\sin \theta_i & \cos \alpha_{(i+1)} \cos \theta_i & -\sin \alpha_{(i+1)} \cos \theta_i \\
0 & \sin \alpha_{(i+1)} & \cos \alpha_{(i+1)}
\end{bmatrix},
$$

(A 5)
Figure 11. Set-up of coordinates and kinematic parameters for (a) zero-thickness and (b) thick-panel origami according to the DH notation.

Table 2. Kinematic behaviour of the general waterbomb tessellation of zero-thickness sheets and thick panels.

<table>
<thead>
<tr>
<th>Geometrical Conditions</th>
<th>Folding Paths</th>
<th>The Waterbomb Tessellation of Zero-thickness Sheets</th>
<th>The Waterbomb Tessellation of Thick Panels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha + \beta &lt; \pi/2$</td>
<td>$\alpha = \beta$</td>
<td>path I: smooth</td>
<td>smooth</td>
</tr>
<tr>
<td></td>
<td></td>
<td>path II: smooth</td>
<td>exists only when $\mu = 1$ and the path is smooth</td>
</tr>
<tr>
<td>$\alpha \neq \beta$</td>
<td></td>
<td>path I: smooth</td>
<td>smooth</td>
</tr>
<tr>
<td></td>
<td></td>
<td>path II: blocked</td>
<td>non-existent</td>
</tr>
<tr>
<td>$\alpha + \beta = \pi/2$</td>
<td>$\alpha = \beta$</td>
<td>path I: two-stage motion</td>
<td>two-stage motion and blocked</td>
</tr>
<tr>
<td></td>
<td></td>
<td>path II: smooth</td>
<td>exists only when $\mu = 1$ and the path is smooth</td>
</tr>
<tr>
<td>$\alpha \neq \beta$</td>
<td></td>
<td>path I: two-stage motion</td>
<td>two-stage motion and blocked</td>
</tr>
<tr>
<td></td>
<td></td>
<td>path II: blocked</td>
<td>non-existent</td>
</tr>
<tr>
<td>$\alpha + \beta &gt; \pi/2$</td>
<td>$\alpha = \beta$</td>
<td>path I: blocked</td>
<td>blocked</td>
</tr>
<tr>
<td></td>
<td></td>
<td>path II: blocked while the path for vertex $W$ is smooth</td>
<td>exists only when $\mu = 1$ but the path is blocked</td>
</tr>
<tr>
<td>$\alpha \neq \beta$</td>
<td></td>
<td>path I: blocked</td>
<td>blocked</td>
</tr>
<tr>
<td></td>
<td></td>
<td>path II: blocked</td>
<td>non-existent</td>
</tr>
</tbody>
</table>

and the inverse transformation is

$$Q_{ii(i+1)} = \begin{bmatrix} \cos \theta_i & \sin \theta_i & 0 \\ -\cos \alpha_{ii(i+1)} \sin \theta_i & \cos \alpha_{ii(i+1)} \cos \theta_i & \sin \alpha_{ii(i+1)} \\ \sin \alpha_{ii(i+1)} \sin \theta_i & -\sin \alpha_{ii(i+1)} \cos \theta_i & \cos \alpha_{ii(i+1)} \end{bmatrix}. \quad (A 6)$$

Equations (A 3) and (A 4) can be used to obtain the closure equations of the thick-panel waterbomb pattern and the original waterbomb origami pattern in the text, respectively.

Appendix B

See table 2.
References


