1 Introduction

The Bricard linkages comprise three deformable octahedrons: the line-symmetric octahedral case, the plane-symmetric octahedral case, and the doubly collapsible octahedral case [1]; and three spatial linkages: the general line-symmetric case, the general plane-symmetric case, and the trihedral case [2]. Baker later pointed out that the line-symmetric octahedral case is a special case of the general line-symmetric linkage case [2,3]. Baker studied all cases of Bricard linkages systematically and derived their closure equations [3]. Later, Phillips reviewed the Bricard linkages and their relationship with other overconstrained linkages [4,5].

The octahedral cases of Bricard linkages attracted a number of kinematic studies. A comprehensive analysis to the three octahedral cases was done by Bennett [6]. Baker noticed the relationship of a special line-symmetric octahedral Bricard linkage in stationary and linkage configurations with respect to the conformation of cyclohexane molecular in chair and boat forms [7]. The closure equations of the three octahedral cases of Bricard linkages were derived analytically using matrix transformation method by Lee [8]. Recently, Chai and Chen found out that the line-symmetric octahedral Bricard linkage with identical twists and offsets always has a stationary structural configuration, which is independent of its mobile linkage form [9]. In engineering applications, the octahedral cases of Bricard linkages are related to parallel manipulators, such as the Stewart-Gough manipulator [10–12] and triangular symmetric simplified manipulators [13,14], which are widely used as flight simulators and milling machines. The independent work by Nelson demonstrated the possibility of building large network of polyhedral with the octahedral cases of Bricard linkages [15].

As for the three linkage cases, a plate-form model of the trihedral Bricard 6R linkage was made and analyzed by Goldberg [16]. Yu [17] studied the geometry of the trihedral case with respect to its circumscribed sphere and associated hyperboloid. Wohlhart’s early work [18] showed that there are two distinct trihedral cases of Bricard linkages. Due to the special geometry constraint of the Bricard linkages, the reciprocal screw system is extensively used for analysis of such mechanisms. Using the reciprocal screw system, it was found that for any configuration of the general line-symmetric Bricard linkage, the central axis of the linear complex defined by the joint axis is orthogonally intersected to the linkage’s line of symmetry [19]. The reciprocal screw system of the general plane-symmetric six-screw linkage was also analyzed by Baker [20], which covers the plane-symmetric case of Bricard linkages. Based on the direct elimination method with optimization theory, Lee [21] proposed a numerical scheme to solve the reciprocal screw system of the Bricard linkages, which gives the necessary solutions to a given linkage, and it is not sufficient to present all possible solutions. Recently, a threefold-symmetric Bricard linkage was proposed to explore the application of Bricard linkage for the design of deployable structures [22]. A special line-symmetric and plane-symmetric Bricard linkages were analyzed with regards to its unique bifurcation behaviours [23]. Further attempt was made to find and identify possible solutions of spatial 6R mechanism with three adjacent parallel axes [24], and the research on the Wohlhart’s symmetric mechanism [25], which is usually identified as the Wohlhart’s hybrid 6R linkage [26,27], explores the industrial application of this linkage as a translator.

To conduct the kinematic study of the general line-symmetric Bricard linkage, the D-H parameters [28] in Fig. 1 are commonly adopted and a homogeneous transformation matrix could be assembled as

\[
\begin{bmatrix}
\cos \theta_i - \cos z_{i+1} \sin \theta_i & \sin z_{i+1} \sin \theta_i & a_{i+1} \cos \theta_i \\
\sin \theta_i \cos z_{i+1} & \cos \theta_i & -a_{i+1} \sin \theta_i \\
0 & 0 & \cos z_{i+1}
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
0 & 0 & 1
\end{bmatrix}
\]  

(1)

Keywords: kinematics, the Bricard linkage, closure equations, multiple closures.
positively equaled on the opposite joints as
variables of the original line-symmetric Bricard linkage are
be the conclusion derived from the closure equations. The
symmetry property in both aspects, but exhibits a line-
geometry conditions of the original general line-
simultaneous Bricard linkage and to explore the
due to the line-symmetric geometry, the revolute
equations of the original line-symmetric Bricard linkage are
closure equations. Here, the entry (1, 1) in Eq. (6) is extracted and reformed as
simplified geometry conditions of the original general line-
i
due to the closure equations. From Eq. (8)
variable, the original and revised line-
Hartenberg’s parameters
Fig. 1 The setup of the Denavit and Hartenberg's parameters
The geometry conditions of the original general line-
Bricard linkage are
Although the closure equations of the general line-
symmetric Bricard linkage have been derived by Baker [3], it requires more elaborations about these equations for a better understanding. Due to the line-symmetric geometry, the revolute variables of the original line-symmetric Bricard linkage are positively equaled on the opposite joints as \( \theta_1 = \theta_2, \theta_3 = \theta_6 \), and \( \theta_5 = \theta_6 \). Yet, a numerical search of new and revised overconstrained linkages was conducted by Mavroidis and Roth [29], where a revised closure of the general line-symmetric Bricard linkage was found with negatively equaled offsets on the opposite joints as below:

It was obtained from numerical method that the revolute variables in the revised general line-symmetric Bricard linkage are negatively equaled on the opposite joints as \( \theta'_1 = -\theta'_4, \theta'_2 = -\theta'_5 \), and \( \theta'_3 = -\theta'_6 \). Despite the similar relationship in geometry conditions and kinematic variables, the original and revised line-symmetric Bricard linkages are usually considered as two distinct linkages.

The original line-symmetric Bricard linkage is with perfect line-symmetry property in both geometric parameters and kinematic variables, while the revised one is without line-symmetry property in both aspects, but exhibits a line-symmetric configuration in the full-cycle motion. Therefore, the focus of this paper is to study the kinematics of the general line-symmetric Bricard linkage and to explore the relationship between the original and revised general line-symmetric Bricard linkages. When deriving the closure equations, only the relationship among the geometric parameters is taken into account. However, the relationship among the kinematic variables is not considered initially. Instead, it will be the conclusion derived from the closure equations. The layout of this paper is as follows. The solutions of closure equations of the original and revised linkages are derived in Secs. 2 and 3, respectively. Section 4 discusses the relationship between the original and revised linkages. Conclusion and further discussions are drawn in Sec. 5, which ends the paper.

2 The Solutions of Closure Equations of the Original General Line-Symmetric Bricard Linkage
The simplified geometry conditions of the original general line-
symmetric Bricard linkage are
\[ a_{i(i+1)} = a_{i(i+3)(i+4)}, \quad a_{i(i+1)} = a_{i(i+3)(i+4)}, \quad R_i = R_{i(i+3)(i+4)}(i = 1, 2, 3) \] (4)
Note that to ensure this is a close-loop 6R mechanism, the sub-
scripts must be the remainder of 6 in positive numbers. The closure condition is
\[ T_{12}T_{23}\cdots T_{61} = I \] (5)
which could be represented as
\[ T_{i(i+1)(i+2)(i+3)} = T_{i(i+3)(i+4)}^{-1}T_{i(i+4)(i+5)}^{-1}T_{i(i+5)(i+4)}^{-1} \] (6)
In the transformation matrix in Eq. (1), the angular parameters, including twist and revolute variable, are stored in both rotational matrix \( R_{i(i+1)} \) and translational vector \( d_{i(i+1)} \). The length parameters, including link length and offset, are only stored in the translational vector \( d_{i(i+1)} \). In Eq. (6), we may first use the rotational matrix \( R_{35} \) to derive the relationship among the angular parameters, and then introduce the length parameters using the translational vector \( d_{35} \) to derive the solutions of closure equations. Here, the entry (1, 1) in Eq. (6) is extracted and reformed as
\[
\begin{align*}
\cos \theta_i \cos \theta_{i+1} \cos \theta_{i+2} - \\
- \cos \theta_{i+1} \cos \theta_{i+4} \cos \theta_{i+5} - \\
\sin \theta_i \sin \theta_{i+1} \cos \theta_{i+2} - \\
- \sin \theta_{i+1} \sin \theta_{i+4} \cos \theta_{i+5} - \\
\cos \theta_i \sin \theta_{i+1} \sin \theta_{i+2} - \\
- \cos \theta_{i+1} \sin \theta_{i+4} \sin \theta_{i+5} - \\
\sin \theta_i \sin \theta_{i+1} \sin \theta_{i+2} - \\
- \sin \theta_{i+1} \sin \theta_{i+4} \sin \theta_{i+5}
\end{align*}
\] (7)
Equation (7) must always hold no matter what the values of twist angles are. Thus, one nontrivial solution is when the items in every bracket of Eq. (7) are zero, i.e.,
\[
\begin{align*}
\cos \theta_i \cos \theta_{i+1} \cos \theta_{i+2} - \cos \theta_{i+3} \cos \theta_{i+4} \cos \theta_{i+5} = 0 \quad (8a) \\
\sin \theta_i \sin \theta_{i+1} \cos \theta_{i+2} - \sin \theta_{i+3} \sin \theta_{i+4} \cos \theta_{i+5} = 0 \quad (8b) \\
\cos \theta_i \sin \theta_{i+1} \sin \theta_{i+2} - \cos \theta_{i+3} \sin \theta_{i+4} \sin \theta_{i+5} = 0 \quad (8c) \\
\sin \theta_i \sin \theta_{i+1} \sin \theta_{i+2} - \sin \theta_{i+3} \sin \theta_{i+4} \sin \theta_{i+5} = 0 \quad (8d) \\
\sin \theta_i \cos \theta_{i+1} \sin \theta_{i+2} - \sin \theta_{i+3} \cos \theta_{i+4} \sin \theta_{i+5} = 0 \quad (8e)
\end{align*}
\]
Substituting Eq. (8d) into Eq. (8e) gives
\[ \cos \theta_{i+1} = \cos \theta_{i+4} \] (9)
From Eq. (8d), we have
\[ \frac{\sin \theta_i}{\sin \theta_{i+3}} = \frac{\sin \theta_{i+5}}{\sin \theta_{i+2}} \] (10)
By substituting $i = 1, 2, 3$ into Eqs. (9) and (10), we have
\begin{equation}
\cos \theta_2 = \cos \theta_5, \quad \cos \theta_3 = \cos \theta_6, \quad \cos \theta_4 = \cos \theta_1
\end{equation}
and
\begin{equation}
\sin \theta_1 \sin \theta_5 = \sin \theta_3 \sin \theta_6 = \sin \theta_4 \sin \theta_2
\end{equation}
Considering the domain of definition that $\theta_i \in [-\pi, \pi]$ and Eqs. (8a)–(8c), the following relationships can be obtained from Eqs. (11) and (12):
\begin{equation}
\text{positive relationship, } \theta_i = \theta_{i+3} \quad (13a)
\end{equation}
\begin{equation}
\text{negative relationship, } \theta_i = -\theta_{i+3} \quad (13b)
\end{equation}
In the above process, entry (1, 1) of the transformation matrix in Eq. (6) is selected for the derivation of the relationships shown as Eq. (13). Alternatively, we can also use entries (2, 2) and (3, 3) in Eq. (6) to derive the same relationships in Eq. (13). Note that these entries are all located in the rotational matrix, which contains only the rotational information during coordinate transformation of the links and joints.
Then, the link length and offset located in the translational vector can be considered to derive the solutions of closure equations. Among the simplified expressions of the translational vectors shown in Eq. (14), entry (3, 4) of the least unknown variables
\begin{equation}
d_1 = T_{(1,4)}(\theta_1, \theta_{i+1}, \theta_{i+2}, \theta_{i+4}, \theta_{i+5}) = 0 \quad (14a)
\end{equation}
\begin{equation}
d_2 = T_{(2,4)}(\theta_1, \theta_{i+1}, \theta_{i+2}, \theta_{i+4}, \theta_{i+5}) = 0 \quad (14b)
\end{equation}
\begin{equation}
d_3 = T_{(3,4)}(\theta_{i+1}, \theta_{i+2}, \theta_{i+4}, \theta_{i+5}) = 0 \quad (14c)
\end{equation}
By substituting the relationship in Eq. (13) into Eq. (14c), for different subscript numbers, we have that
\begin{itemize}
\item when $i = 1$, the relationship between $\theta_2$ and $\theta_4$ can be derived
\item when $i = 2$, the relationship between $\theta_3$ and $\theta_4$ can be derived
\item when $i = 3$, the relationship between $\theta_1$ and $\theta_2$ can be derived
\end{itemize}
To form the closure equations, $\theta_1$ is taken as the input. Then, only the relationship between $\theta_1$ and $\theta_{i+3}$ will be obtained in the following process. Together with Eq. (13), the complete set of solutions to the closure equations for the original general line-symmetric Bricard 6R linkage will be obtained.

### 2.1 Positive Relationship: $\theta_i = \theta_{i+3}$
Firstly, we consider the case of positive relationship in Eq. (13a) that $\theta_i = \theta_{i+3}$, where the revolute variables follow the property of line-symmetry. When $i = 3$, substituting Eq. (13a) into Eq. (14c) gives
\begin{equation}
(a_{12} \sin \alpha_{12} + a_{34} \sin \alpha_{12} \cos \alpha_{23}) \sin \theta_1
- (R_2 \sin \alpha_{12} \sin \alpha_{34} + R_3 \sin \alpha_{12} \cos \alpha_{23} \sin \alpha_{34}) \cos \theta_1
+ (a_{12} \cos \alpha_{23} + a_{23} \cos \alpha_{23} \sin \alpha_{34}) \sin \theta_2
- (R_3 \sin \alpha_{12} \cos \alpha_{34} \sin \alpha_{23} \cos \alpha_{34}) \cos \theta_2
+ R_3 \sin \alpha_{23} \cos \alpha_{34} \sin \theta_1 \cos \theta_2
+ (a_{12} \cos \alpha_{23} \sin \alpha_{23} + a_{23} \sin \alpha_{23} \cos \alpha_{34}) \sin \theta_1 \cos \theta_2
+ (a_{23} \cos \alpha_{23} \sin \alpha_{34} + a_{34} \cos \alpha_{23} \sin \alpha_{34}) \cos \theta_1 \sin \theta_2
- R_3 \cos \alpha_{12} \sin \alpha_{34} \cos \alpha_{23} \cos \alpha_{34} \cos \theta_2
+ R_1 \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{34}
+ R_2 \cos \alpha_{23} \cos \alpha_{23} \cos \alpha_{34}
+ R_3 (1 + \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{34}) = 0
\end{equation}

The above equation can be simplified as
\begin{equation}
A_2 \sin \theta_1 + B_2 \cos \theta_1 + C_2 \sin \theta_1 + D_2 \cos \theta_1
+ E_2 \sin \theta_1 + F_2 \sin \theta_1 \cos \theta_2 + G_2 \cos \theta_1 \sin \theta_2
+ H_2 \cos \theta_1 \cos \theta_2 + L_2 = 0
\end{equation}
in which
\begin{align*}
A_2 &= + (a_{12} \sin \alpha_{12} + a_{34} \sin \alpha_{12} \cos \alpha_{23}) \\
B_2 &= - (R_2 \sin \alpha_{12} \sin \alpha_{34} + R_3 \sin \alpha_{12} \cos \alpha_{23} \sin \alpha_{34}) \\
C_2 &= + (a_{12} \cos \alpha_{23} + a_{23} \cos \alpha_{23} \sin \alpha_{34}) \\
D_2 &= - (R_1 \sin \alpha_{12} \sin \alpha_{23} + R_3 \sin \alpha_{12} \cos \alpha_{23} \cos \alpha_{34}) \\
E_2 &= + R_3 \sin \alpha_{23} \sin \alpha_{34} \\
F_2 &= + (a_{12} \cos \alpha_{12} \sin \alpha_{23} + a_{23} \cos \alpha_{23} \sin \alpha_{34}) \\
G_2 &= + (a_{12} \cos \alpha_{12} \cos \alpha_{34} + a_{34} \cos \alpha_{23} \sin \alpha_{34}) \\
H_2 &= - R_1 \cos \alpha_{12} \sin \alpha_{23} \cos \alpha_{34} \\
I_2 &= + R_1 \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{34}
+ R_2 (\cos \alpha_{23} + \cos \alpha_{12} \cos \alpha_{34})
+ R_3 (1 + \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{34})
\end{align*}
After the tangent half-angle substitution of sin $\theta_2$ and cos $\theta_2$, Eq. (16) can be rewritten into
\begin{equation}
\left[ (A_2 \sin \theta_1 + B_2 \cos \theta_1 + L_2) \right] \tan^2 \frac{\theta_2}{2}
\end{equation}
\begin{equation}
+ 2(C_2 + E_2 \sin \theta_1 + F_2 \cos \theta_1) \tan \frac{\theta_2}{2} + (A_2 \sin \theta_1 + B_2 \cos \theta_1 + L_2)
\end{equation}
\begin{equation}
+ (D_2 + F_2 \sin \theta_1 + H_2 \cos \theta_1) \tan \frac{\theta_2}{2}
= 0
\end{equation}
Again, Eq. (18) can be further simplified as
\begin{equation}
Aterm_2 \tan^2 \frac{\theta_2}{2} + Bterm_2 \tan \frac{\theta_2}{2} + Cterm_2 = 0
\end{equation}
in which $\theta_1$ is represented in
\begin{equation}
Aterm_2 = (A_2 \sin \theta_1 + B_2 \cos \theta_1 + L_2)
\end{equation}
\begin{equation}
Bterm_2 = 2(C_2 + E_2 \sin \theta_1 + F_2 \cos \theta_1)
\end{equation}
\begin{equation}
Cterm_2 = (A_2 \sin \theta_1 + B_2 \cos \theta_1 + L_2)
+ (D_2 + F_2 \sin \theta_1 + H_2 \cos \theta_1)
\end{equation}
Solutions to Eq. (19) are
\begin{equation}
\frac{\theta_2}{2} = - Bterm_2 \pm \sqrt{Bterm_2^2 - 4Aterm_2 \cdot Cterm_2}
\end{equation}
in which all symbols are defined in Eqs. (17) and (20). The closure relationship between $\theta_1$ and $\theta_3$ is therefore obtained. Similarly, by analysing Eqs. (13a) and (14c), with $i = 2$, the relationship between $\theta_1$ and $\theta_6$ can be derived as
\begin{equation}
\frac{\theta_3}{2} = - Bterm_3 \pm \sqrt{Bterm_3^2 - 4Aterm_3 \cdot Cterm_3}
\end{equation}
where

\[
\begin{align*}
A_{term} &= (A_3 \sin \theta_1 + B_3 \cos \theta_1 + L_3) \\
-B_{term} &= 2(C_3 + E_3 \sin \theta_1 + G_3 \cos \theta_1) \\
C_{term} &= (A_3 \sin \theta_1 + B_3 \cos \theta_1 + L_3) \\
+(D_3 + F_3 \sin \theta_1 + H_3 \cos \theta_1)
\end{align*}
\]

and

\[
\begin{align*}
A_3 &= +(a_{12} \cos z_{23} \sin z_{34} + a_{34} \sin z_{12}) \\
B_3 &= -(R_3 \sin z_{12} \cos z_{23} + R_3 \sin z_{12} \sin z_{34}) \\
C_3 &= +(a_{23} \cos z_{12} \sin z_{34}) \\
D_3 &= -(R_3 \cos z_{12} \sin z_{23} + R_3 \sin z_{21} \cos z_{34}) \\
E_3 &= +(R_3 \sin z_{12} \sin z_{23}) \\
F_3 &= +(a_{12} \sin z_{23} \cos z_{34} + a_{34} \sin z_{12}) \\
G_1 &= +(a_{23} \sin z_{12} \cos z_{34} + a_{12} \sin z_{23}) \\
H_1 &= -(R_3 \sin z_{12} \sin z_{23} \cos z_{34}) \\
I_1 &= +R_3 \cos z_{12} \sin z_{23} + \cos z_{34} \\
+R_3(1 + \cos z_{12} \cos z_{23} \cos z_{34}) \\
+R_3(\cos z_{23} + \cos z_{12} \cos z_{34})
\end{align*}
\]

Because not all of the entries in the transformation matrix in Eq. (6) are included in the derivation process, the above results only demonstrate the necessary conditions for the closure relationship among the revolute variables. To ensure the closure, Eqs. (13a), (21), and (22) have been taken into the transformation matrix to check whether Eq. (5) is held for the general geometric parameters. In such a way, two sets of solutions to the closure equations are obtained to achieve different linkage closures as follows:

\[
\begin{align*}
\theta_2 &= 2 \tan^{-1} \left( \frac{-B_{term} + \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}} \right) \\
\theta_3 &= 2 \tan^{-1} \left( \frac{-B_{term} - \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}} \right) \\
\theta_4 &= \theta_1 \\
\theta_5 &= \theta_2 \\
\theta_6 &= \theta_3
\end{align*}
\]

and

\[
\begin{align*}
\theta_2 &= 2 \tan^{-1} \left( \frac{-B_{term} + \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}} \right) \\
\theta_3 &= 2 \tan^{-1} \left( \frac{-B_{term} - \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}} \right) \\
\theta_4 &= \theta_1 \\
\theta_5 &= \theta_2 \\
\theta_6 &= \theta_3
\end{align*}
\]

Results shown in Eqs. (25) and (26) indicate that there are two distinct forms of the original general line-symmetric Bricard 6R linkage, both with the property of line-symmetry, named as Form I linkage and Form II linkage, respectively. Their kinematic paths are plotted in Figs. 2 and 3. Their spatial configurations are
illustrated in Figs. 4 and 5, in which the lines of symmetry are identified as the central lines in front views and dashed circles in top views. The geometry parameters of the original general line-symmetric Bricard linkages in Figs. 2–5 are set as

\[
\begin{align*}
 a_{12} &= 2.40, & a_{23} &= 2.90, & a_{34} &= 1.50 \\
 x_{12} &= 4\pi/18, & x_{23} &= 8\pi/18, & x_{34} &= 13\pi/18 \\
 R_1 &= 0.50, & R_2 &= 0.55, & R_3 &= 0.42
\end{align*}
\]

The singularity behaviours of these two linkage forms are examined with the singular value decomposition method, which is a numerical method to solve the singular values of the linkage’s Jacobian matrix. It is found that these two kinematic paths are solely existed without any bifurcation points, see Fig. 6. Therefore, from Figs. 2, 3, and 6, we can conclude that these two linkage forms are independent with no common configurations under the same geometry conditions.

2.2 Negative Relationship: \( \theta_i = -\theta_{i+3} \). For the case of negative relationship, the revolute variables do not follow the line-symmetry property. We can follow the same procedure as the positive relationship to derive the solutions to the closure equations. However, when substituting the results into the transformation matrix, the closure condition in Eq. (5) is not held. Thus, no closed linkage could be achieved with \( \theta_i = -\theta_{i+3} \), which means the negative relationship of \( \theta_i = -\theta_{i+3} \) is untrue for the original general line-symmetric Bricard linkage.

3 The Solutions of Closure Equations of the Revised General Line-Symmetric Bricard Linkage

The simplified geometry conditions of the revised general line-symmetric Bricard linkage are

\[
\begin{align*}
 a'_{i(i+1)} &= a'_{(i+3)(i+4)}, & a'_{(i+1)} &= a'_{(i+3)(i+4)} \\
 R'_i &= -R'_{i+3}(i = 1, 2, 3)
\end{align*}
\]

The only difference between the geometry conditions of the original and revised linkages is the offset conditions. And entry (1, 1) is in the rotational matrix. Thus, Eq. (7) applies for both the original and revised linkages. Therefore, Eqs. (13a) and (13b) also can be obtained for the revised general line-symmetric Bricard linkage.

3.1 Positive Relationship: \( \theta'_i = \theta'_{i+3} \). For the case of positive relationship, the revolute variables follow the line-symmetry
property. We can follow the same procedure as Sec. 2.1 to derive the solutions to the closure equations. However, when substituting the result into the transformation matrix, the closure condition in Eq. (5) is not held. Thus, no closed linkage could be achieved with $h_i = h_i + 3$, which means the positive relationship of $h_i = h_i + 3$ is untrue for the revised general line-symmetric Bricard linkage.

### 3.2 Negative Relationship: $\theta'_i = -\theta'_{i-3}$

The same procedure could be carried out to derive the solutions of closure equations for the case of negative relationship. As a result, the following two sets of solutions to the closure equations are concluded to produce two different linkage closures, which are named as the Form I and Form II of the revised general line-symmetric Bricard linkages:

$$
\begin{align*}
\theta'_2 &= 2 \tan^{-1} \left( \frac{-Bterm'_2 + \sqrt{Bterm'_2^2 - 4Aterm'_2 \cdot Cterm'_2}}{2Aterm'_2} \right) \\
\theta'_3 &= 2 \tan^{-1} \left( \frac{-Bterm'_3 - \sqrt{Bterm'_3^2 - 4Aterm'_3 \cdot Cterm'_3}}{2Aterm'_3} \right) \\
\theta'_4 &= -\theta'_1 \\
\theta'_5 &= -\theta'_2 \\
\theta'_6 &= -\theta'_3 \\
\end{align*}
$$

Fig. 6 The SVD results of the original general line-symmetric Bricard $6R$ linkages: (a) Form I linkage and (b) Form II linkage

Fig. 7 The kinematic paths of the revised Form I general line-symmetric Bricard linkage

and

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\[
\begin{align*}
\theta'_2 &= 2 \tan^{-1}\left(\frac{-B_{term}'_2 - \sqrt{B_{term}'_2^2 - 4A_{term}'_2 \cdot C_{term}'_2}}{2A_{term}'_2}\right) \\
\theta'_3 &= 2 \tan^{-1}\left(\frac{-B_{term}'_3 + \sqrt{B_{term}'_3^2 - 4A_{term}'_3 \cdot C_{term}'_3}}{2A_{term}'_3}\right) \\
\theta'_4 &= -\theta'_1 \\
\theta'_5 &= -\theta'_3 \\
\theta'_6 &= -\theta'_4 \\
\end{align*}
\]

where

\[
\begin{align*}
A_{term}'_1 &= \left( A'_2 \sin \theta'_1 + B'_2 \cos \theta'_1 + L'_2 \right) \\
B_{term}'_2 &= 2\left( C'_3 + E'_2 \sin \theta'_1 + G'_2 \cos \theta'_1 \right) \\
C_{term}'_2 &= \left( A'_2 \sin \theta'_1 + B'_2 \cos \theta'_1 + L'_2 \right) \\
D_{term}'_2 &= \left( D'_3 + F'_2 \sin \theta'_1 + H'_2 \cos \theta'_1 \right) \\
A_{term}'_3 &= \left( A'_4 \sin \theta'_3 + B'_4 \cos \theta'_3 + L'_4 \right) \\
B_{term}'_3 &= 2\left( C'_5 + E'_4 \sin \theta'_3 + G'_4 \cos \theta'_3 \right) \\
C_{term}'_3 &= \left( A'_4 \sin \theta'_3 + B'_4 \cos \theta'_3 + L'_4 \right) \\
D_{term}'_3 &= \left( D'_5 + F'_4 \sin \theta'_3 + H'_4 \cos \theta'_3 \right) \\
\end{align*}
\]

\[
\begin{align*}
\theta'_1 &= \left( a'_{14} \sin x'_{12} \cos x'_{23} - a'_{12} \sin x'_{34} \right) \\
\theta'_2 &= \left( R'_1 \sin x'_{12} \cos x'_{23} + R'_2 \sin x'_{12} \sin x'_{34} \right) \\
\theta'_3 &= \left( a'_{23} \sin x'_{12} \cos x'_{23} - a'_{24} \sin x'_{23} \right) \\
\theta'_4 &= \left( R'_3 \sin x'_{12} \cos x'_{23} - R'_4 \sin x'_{12} \sin x'_{34} \right) \\
\theta'_5 &= \left( 1 - \cos x'_{12} \cos x'_{23} \cos x'_{34} \right) \\
\theta'_6 &= \left( 1 - \cos x'_{12} \cos x'_{23} \cos x'_{34} \right)
\end{align*}
\]

The following geometry conditions are used to plot the kinematic paths in Figs. 7 and 8 using Eqs. (29) and (30), respectively. The spatial configurations of these two linkage forms are plotted in Figs. 9 and 10, respectively. Note that the twists on links 34 and 61 in Eq. (27) differ from the twists on links 3’4’ and 6’1’ in Eq. (35) by \( \pi \). And the offsets are negatively equaled in Eq. (35c).

\[
\begin{align*}
ad'_{12,45} &= 2.40, \quad ad'_{23,56} = 2.90, \quad ad'_{34,61} = 1.50 \\
x'_{12,45} &= 4\pi/18, \quad x'_{23,56} = 8\pi/18, \quad x'_{34,61} = -5\pi/18 \\
R'_1 &= -R'_4 = 0.50, \quad R'_2 = -R'_3 = 0.55, \quad R'_6 = 0.42
\end{align*}
\]

Fig. 8 The kinematic paths of the revised Form II’ general line-symmetric Bricard 6R linkage.
4 Relationship Between the Original and Revised General Line-Symmetric Bricard Linkages

By comparing the solutions of closure equations of the original general line-symmetric Bricard linkage in Eqs. (25) and (26) with the revised general line-symmetric Bricard linkage’s in Eqs. (29) and (30), it can be found that when

\[
\begin{align*}
    a_{12} &= a_{01} = a_{45} = a_{45}' = a_{56} = a_{56}', \\
    a_{34} &= a_{34}' = a_{03} = a_{03}', \\
    x_{12} &= x_{12}' = x_{45} = x_{45}' = x_{56} = x_{56}', \\
    x_{34} &= x_{34}' = x_{03}' = x_{03}', \\
    R_1 &= R_1' = R_4 = -R_4', \\
    R_2 &= R_2' = R_5 = -R_5', \\
    R_3 &= R_3' = R_6 = -R_6'
\end{align*}
\] (36)

we will have

\[
\begin{align*}
    \theta_1 &= \theta_1', \\
    \theta_2 &= \theta_2', \\
    \theta_3 &= \theta_3', \\
    \theta_4 &= -\theta_4', \\
    \theta_5 &= -\theta_5', \\
    \theta_6 &= -\theta_6'
\end{align*}
\] (37)

for both linkage forms, which has been confirmed by the kinematic paths in Figs. 2, 3, 7, and 8. Even though the geometry
conditions and revolute variables in the revised general line-symmetric Bricard linkage are not line-symmetric, the spatial configurations of the resultant linkages in Figs. 9 and 10 are still in a line-symmetric manner.

In fact, with the geometric parameters in Eqs. (27) and (35) which satisfies Eq. (36), the spatial configurations of the revised Form I and II linkages in Figs. 9 and 10 are the same as those of the original Form I and II linkages in Figs. 4 and 5. Take the Form I of the original and revised general line-symmetric Bricard linkages, for example, the joints 4, 5, 6 in Figs. 11(a) and 11(b) are in opposite directions due to as4 = as4 + π = as6 = as6 + π. As a result, θ4 = −θ1, θ5 = −θ2, θ6 = −θ3. The same analysis could be carried out for the relationship between the Form II original and revised linkages. The original and revised linkages are actually equivalent to each other with different setups on joint axis directions.

5 Conclusion and Discussions

In this paper, the kinematics of the original general line-symmetric Bricard 6R linkage is investigated through the algebraic derivation of the solutions to the closure equations. It is found that there are two independent linkage forms, which are called the Form I linkage and Form II linkage, under the same geometry conditions. The revised general line-symmetric Bricard linkage is also investigated with negatively equaled offsets on the opposite joints. Further analysis shows that the original and revised linkages are equivalent with different setups on the joint axis directions. Results in this paper offer an in-depth understanding about the kinematics of the general line-symmetric Bricard linkage.

For the general line-symmetric octahedral Bricard linkage as a special case of the general line-symmetric Bricard linkage, we can substitute ai,i+1 = 0 (i = 1, 2, …, 6) into the two sets of closure equations of the general line-symmetric Bricard linkage to give the closure equations of the line-symmetric octahedral Bricard linkage. As shown in Fig. 12, it is found that the closure equations of each linkage form can only produce half of the kinematic paths of the line-symmetric octahedral Bricard linkage, which then join together at points P1 and P2 to form a full-cycle motion of the linkage. When the revolute variables are in negative relationship, no closure can be achieved for the line-symmetric octahedral Bricard linkage. The results shown in Fig. 12 comply with previous results in Refs. [8,9].

It should be pointed out that since the quadratic equation is used in the process to solve the closure equations, its discriminant must be non-negative in order to obtain the valid solution. For example, the discriminant of Eq. (19) must follow the condition to ensure a valid input–output relationship between θ1 and θ2

$$B_{term}^2 - 4A_{term} \times C_{term} \geq 0 \quad (38)$$

and for the relationship between θ1 and θ3

$$B_{term}^3 - 4A_{term} \times C_{term} \geq 0 \quad (39)$$

must hold. Equations (38) and (39) contain only the input kinematic variable θ1 and the geometric parameters of the linkage, ai,i+1, zi,i+1, and Ri. Then the combinations of their solutions for the range of θ1 in the terms of ai,i+1, zi,i+1, and Ri offer the valid linkage closure between input kinematic variable θ1 and the output kinematic variables θ2, 3, 4, 5, 6, which is established in Eqs. (25) and (26). So with the given geometric parameters of the linkage, the valid input kinematic variable can be determined by Eqs. (38) and (39). In general, the line-symmetric Bricard linkage in Figs. 2, 3, 7, and 8, θi ∈ [−π, π], and the special line-symmetric octahedral Bricard linkage in Fig. 12, the domain of the definition for θ1 is [−2.4816, 2.4914].

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Nomenclature

| A, B, ..., H, and L | the symbols of simplified mathematical relationships |
| a(i,i+1) | the length of link (i + 1), which is the common normal distance from zi to zi+1 positively about xi, and defined in the range of (−∞, +∞) |
| Aterm, Bterm, and Cterm | the symbols of simplified mathematical relationships |
| d(i,i+1) | the 3 × 1 translational vector |
| Forms I, II | the different linkage closures |
| I | the identity matrix |
| Ri | the offset of joint i, which is the common normal distance from xi to x(i+1) positively along zi, and defined in the range of (−∞, +∞) |
| R3×3 | the 3 × 3 rotational matrix |
| T(i,i+1) | the transformation matrix from joint i to joint i + 1 |
| x and x’ | the corresponding parameters in two types of general line-symmetric Bricard linkage, where x is for the parameters in the original linkage and x’ is for the parameters in the revised linkage |
| xi | the coordinate axis along the common normal between joint axes from joint i to joint i + 1 |
\[ z_i = \text{the coordinate axis along the revolute axis of joints } i \]

\[ \theta_i = \text{the twist of link } i(i+1), \text{ which is the rotation angle from } z_i \text{ to } z_{i+1} \text{ positively about } x_i \text{ and defined in the range of } [-\pi, \pi] \]

\[ \phi_i = \text{the revolute variable of joint } i, \text{ which is the rotation angle from } x_i \text{ to } x_{i+1} \text{ positively about } z_i \text{ and defined in the range of } [-\pi, \pi] \]

References


