A 6R linkage reconfigurable between the line-symmetric Bricard linkage and the Bennett linkage

C.Y. Songa, Yan Chenb,⁎, I-Ming Chena

aSchool of Mechanical and Aerospace Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore
bKey Laboratory of Mechanism Theory and Equipment Design of Ministry of Education, Tianjin University, Tianjin 300072, China

Article history:
Received 26 May 2013
Received in revised form 15 July 2013
Accepted 17 July 2013
Available online 22 August 2013

Keywords:
Line-symmetric Bricard linkage
Bennett linkage
Spatial triangle
Reconfigurable mechanism
Bifurcation analysis

1. Introduction

There are two major linkage families among the various single-loop overconstrained spatial linkages: the Bennett-based one [1] and the Bricard-related one. In the former one, the Bennett linkage [2,3] is used as the basic construct unit to form different overconstrained 5R and 6R linkages, such as Myard linkages [4], extended Myard linkage [5], Goldberg’s 5R and 6R linkages [6], Waldron’s hybrid 6R linkages [7], Yu and Baker’s syncopated 6R linkage [8], generalised Goldberg 5R linkage and Wohlhart’s double-Goldberg 6R linkage [9], mixed double-Goldberg 6R linkages [10] and so on. Linkages in the Bricard-related family usually contain certain symmetric geometry properties to enable mobility [11,12], including three octahedral cases [13], three linkage cases [14], Altmann’s 6R linkage [15], Wohlhart’s hybrid 6R linkage [16] and so on. However, there is little interaction between the linkages in Bennett-based linkage family and those in Bricard-related one, except for the isomerization introduced by Wohlhart [17], which sets up the connection between the Wohlhart’s double-Goldberg 6R linkage and the line-symmetric Bricard linkage.

Recent development in mechanism and machine design promotes the concept of reconfigurable mechanism, which has emerged into three major categories. The first is based on the re-assembly of identical or similar robotics modules [18–20], each of which is an integrated system of microprocessors, batteries, sensors, end-effectors, etc. The second is the metamorphic mechanism [21–23], which can generate different topologies for reconfigurations. The third is to incorporate certain bifurcation behaviours to the existing linkage’s kinematic paths [24–26]. At the transit configurations, kinematotropy mechanism [25,27] can change its global mobility. The mechanism reported by Kong and Huang [24] can change between two operation modes. And the multifunctional 7R mechanism can function as two different types of overconstrained linkages [26]. A comprehensive review about the current development, principles and strategies of the reconfigurable mechanisms was discussed in [28]. The potential of reconfiguration can be identified when two or more subgroups are involved in the construct of the mechanism [29]. The
bifurcation behaviour of the double-subtractive-Goldberg 6R linkage was analysed in [30], where multiple linkage closures were found using both construct method and non-construct method. As the result, this Goldberg 6R linkage could be reconfigured among four different 6R linkages, whose kinematic curves form a closed loop through four bifurcation points. Recently, the Wohlhart’s double-Goldberg 6R linkage was analysed and the operation form of a 4R linkage is successfully introduced to the linkage’s bifurcation paths[31]. Here, the effort is made to construct reconfigurable mechanism under the third category.

In this paper, two spatial triangles and Bennett linkages are used as the basic elements to construct the 6R linkages with different reconfiguration capabilities. In Section 2, the kinematics of the spatial triangle, the Bennett linkage and the general line-symmetric Bricard linkage are introduced. Sections 3 and 4 demonstrate the construct process of an asymmetric 6R linkage and a line-symmetric 6R linkage, as well as analyse their kinematic bifurcation for reconfiguration. Final conclusions are drawn in Section 5.
2. Preliminaries

2.1. The spatial triangle structure

The spatial triangle is a single-loop structure enclosed by three spatial links and connected by three revolute joints. It was initially introduced and analysed by Yang [32] using dual quaternion method. Later, Mavroidis and Roth [33] studied the kinematics of the spatial polygons using matrix method and extended the research into screw polygons. And screw triangles were applied to unify the finite and infinitesimal kinematics [34]. The importance of spatial triangle was recently revisited by Huang [35].

In Fig. 1, a spatial triangle is defined with DH parameters [36].

With the geometry conditions of links 12 and 23 in the spatial triangle, i.e. $a_{12}$, $\alpha_{12}$, $a_{23}$, $\alpha_{23}$, $R_2$ and $\theta_2$, using the closure matrix condition that

$$T_{12}T_{23}T_{31} = I,$$  

(1)

the geometric parameters on link 31 can be derived as

$$a_{31} = a_{23}(\cos \alpha_{12} \sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2) - a_{12} \cos \theta_1 - R_2 \sin \alpha_{12} \sin \theta_1,$$

$$\tan \alpha_{31} = -\frac{\sin \theta_1 \cos \theta_2 + \cos \alpha_{12} \sin \theta_2 \cos \theta_1}{\sin \alpha_{12} \sin \theta_2},$$

$$R_1 = \frac{a_{12}(\cos \alpha_{34} \sin \theta_1 \sin \theta_3 - \cos \theta_1 \cos \theta_3) - a_{23} - a_{31} \cos \theta_3}{\sin \alpha_{34} \sin \theta_3},$$

$$R_3 = \frac{a_{31}(\cos \alpha_{23} \sin \theta_2 \sin \theta_3 - \cos \theta_2 \cos \theta_3) - a_{12} - a_{23} \cos \theta_2}{\sin \alpha_{23} \sin \theta_3},$$

$$\tan \theta_1 = -\frac{\sin \alpha_{12} \cos \alpha_{12} \sin \alpha_{23} \sin \alpha_{23} \cos \theta_1}{\sin \alpha_{12} \sin \alpha_{23} \sin \theta_2},$$

$$\tan \theta_3 = -\frac{\sin \alpha_{23} \cos \alpha_{12} \sin \alpha_{23} \sin \alpha_{12} \cos \theta_2}{2}.$$  

(2)

2.2. The Bennett linkage with two different setups

The original setup of the Bennett linkage [2,3] is shown in Fig. 2(a). Its geometry conditions and closure equations are

$$a_{12} = a_{34}, \alpha_{12} = \alpha_{34}, a_{23} = a_{41}, \alpha_{23} = \alpha_{41}, R_i = 0 (i = 1, 2, 3 \text{ and } 4),$$

(3a)

$$\sin \frac{\alpha_{12}}{a_{12}} = \frac{\sin \alpha_{23}}{a_{23}};$$

(3b)

and

$$\theta_1 + \theta_3 = 0,$$

(4a)

$$\theta_2 + \theta_4 = 0,$$

(4b)

$$\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{\sin \alpha_{23} + \alpha_{12}}{\sin \alpha_{23} - \alpha_{12}}.$$  

(4c)

respectively.

The proportional relationship of the sine of twist over link length is called the Bennett ratio, as shown in Eq. (3b). The geometry conditions in Eqs. (3a) and (3b) are typically in a line-symmetric form. But the closure equations in Eqs. (4a) and (4b) do not follow the line-symmetric condition. This is because the revolute axes of the Bennett linkage in Fig. 2(a) are not set up in a
line-symmetric manner. As shown in Fig. 2(b), by reversing the axes of joints 3 and 4, the resultant linkage becomes a Bennett linkage in a line-symmetric setup [37,38]. The corresponding geometry conditions and closure equations are

\[ a_{12} = a_{34}, \alpha_{12} = \alpha_{34}, a_{23} = a_{41}, \alpha_{23} = \alpha_{41}, R_i = 0 (i = 1, 2, 3 \text{ and } 4), \]

\[ \sin \frac{\alpha_{12}}{a_{12}} = -\sin \frac{\alpha_{23}}{a_{23}}, \]

and

\[ \theta_1 = \theta_3, \theta_2 = \theta_4, \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{\cos \frac{\alpha_{23} + \alpha_{12}}{2}}{\cos \frac{\alpha_{23} - \alpha_{12}}{2}}. \]

respectively.

2.3. The general line-symmetric Bricard linkage

The geometry conditions of the general line-symmetric Bricard linkage [14,39] are

\[ a_{12} = a_{45}, \ a_{23} = a_{56}, \ a_{34} = a_{61}, \]
\[ \alpha_{12} = \alpha_{45}, \ \alpha_{23} = \alpha_{56}, \ \alpha_{34} = \alpha_{61} , \]
\[ R_1 = R_4, \ R_2 = R_5, \ R_3 = R_6. \]

The explicit closure equations of the linkage have been derived in [40] and two closures of general line-symmetric Bricard linkage forms are found with the identical geometry conditions, whose explicit closure equations are

\[ \begin{align*}
\theta_2 &= 2 \tan^{-1} \left( \frac{-B_{term} + \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}} \right) \\
\theta_3 &= 2 \tan^{-1} \left( \frac{-B_{term} - \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}} \right) \quad (8)
\end{align*} \]

\[ \begin{align*}
\theta_4 &= \theta_1 \\
\theta_5 &= \theta_2 \\
\theta_6 &= \theta_3
\end{align*} \]

Fig. 2. The two different joint-axis setups of Bennett linkage: (a) in the asymmetric setup; (b) in the line-symmetric setup.
And

\[
\begin{align*}
\theta_2 &= 2 \tan^{-1}\left( \frac{-B_{\text{term}2} - \sqrt{B_{\text{term}2}^2 - 4A_{\text{term}2} \cdot C_{\text{term}2}}}{2A_{\text{term}2}} \right), \\
\theta_3 &= 2 \tan^{-1}\left( \frac{-B_{\text{term}3} + \sqrt{B_{\text{term}3}^2 - 4A_{\text{term}3} \cdot C_{\text{term}3}}}{2A_{\text{term}3}} \right).
\end{align*}
\]

\[ \theta_4 = \theta_1, \quad \theta_5 = \theta_2, \quad \theta_6 = \theta_3 \]

respectively. All symbols are defined as follows.

\[
\begin{align*}
A_{\text{term}2} &= (A_3 \sin \theta_1 + B_3 \cos \theta_1 + L_2) - (D_2 + F_2 \sin \theta_1 + H_2 \cos \theta_1), \\
B_{\text{term}2} &= 2(C_2 + E_2 \sin \theta_1 + G_2 \cos \theta_1), \\
C_{\text{term}2} &= (A_2 \sin \theta_1 + B_2 \cos \theta_1 + L_2) + (D_2 + F_2 \sin \theta_1 + H_2 \cos \theta_1), \quad (9)
\end{align*}
\]

\[
\begin{align*}
A_{\text{term}3} &= (A_3 \sin \theta_1 + B_3 \cos \theta_1 + L_3) - (D_3 + F_3 \sin \theta_1 + H_3 \cos \theta_1), \\
B_{\text{term}3} &= 2(C_3 + E_3 \sin \theta_1 + G_3 \cos \theta_1), \\
C_{\text{term}3} &= (A_3 \sin \theta_1 + B_3 \cos \theta_1 + L_3) + (D_3 + F_3 \sin \theta_1 + H_3 \cos \theta_1), \quad (10)
\end{align*}
\]

\[
\begin{align*}
A_1 &= + (a_{34} \sin \alpha_{12} \cos \alpha_{23} + a_{13} \sin \alpha_{34}) \\
B_2 &= + (a_{34} \sin \alpha_{12} \cos \alpha_{23} + a_{13} \sin \alpha_{34}) \\
C_2 &= + (a_{23} \sin \alpha_{12} \cos \alpha_{23} + a_{12} \sin \alpha_{23}) \\
D_2 &= + (a_{23} \sin \alpha_{12} \cos \alpha_{23} + a_{12} \sin \alpha_{23}) \\
E_2 &= + (a_{23} \sin \alpha_{12} \cos \alpha_{23} + a_{12} \sin \alpha_{23}) \\
F_3 &= + (a_{23} \sin \alpha_{12} \cos \alpha_{23} + a_{12} \sin \alpha_{23}) \\
G_3 &= + (a_{23} \sin \alpha_{12} \cos \alpha_{23} + a_{12} \sin \alpha_{23}) \\
H_3 &= + (a_{23} \sin \alpha_{12} \cos \alpha_{23} + a_{12} \sin \alpha_{23}) \\
L_2 &= + (R_1 (\cos \alpha_{12} \cos \alpha_{23} + \cos \alpha_{34}) + R_2 (\cos \alpha_{12} \cos \alpha_{23} + \cos \alpha_{34}) \nonumber \\
&\quad + R_3 (1 + \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{34}). \quad (12)
\end{align*}
\]

\[
\begin{align*}
A_3 &= + (a_{12} \cos \alpha_{23} \sin \alpha_{34} + a_{34} \sin \alpha_{12}) \\
B_3 &= + (a_{12} \cos \alpha_{23} \sin \alpha_{34} + a_{34} \sin \alpha_{12}) \\
C_3 &= + (a_{23} \sin \alpha_{12} \sin \alpha_{23} + a_{13} \sin \alpha_{23}) \\
D_3 &= + (a_{23} \sin \alpha_{12} \sin \alpha_{23} + a_{13} \sin \alpha_{23}) \\
E_3 &= + (a_{23} \sin \alpha_{12} \sin \alpha_{23} + a_{13} \sin \alpha_{23}) \\
F_3 &= + (a_{23} \sin \alpha_{12} \sin \alpha_{23} + a_{13} \sin \alpha_{23}) \\
G_3 &= + (a_{23} \sin \alpha_{12} \sin \alpha_{23} + a_{13} \sin \alpha_{23}) \\
H_3 &= + (a_{23} \sin \alpha_{12} \sin \alpha_{23} \sin \alpha_{34}) \\
L_3 &= + (R_1 (\cos \alpha_{12} + \cos \alpha_{23} \cos \alpha_{34}) + R_2 (\cos \alpha_{12} + \cos \alpha_{23} \cos \alpha_{34}) \nonumber \\
&\quad + R_3 (\cos \alpha_{23} \cos \alpha_{12} \cos \alpha_{34}). \quad (13)
\end{align*}
\]

3. The asymmetric 6R linkage

Two identical spatial triangles 123 and 456 are prepared for construction in Fig. 3, whose geometry conditions are set up as follows.

\[
\begin{align*}
\alpha_{12}^{ST} &= \alpha_{12}^{ST}, \quad \alpha_{13}^{ST} = \alpha_{13}^{ST}, \quad \alpha_{23}^{ST} = \alpha_{23}^{ST}, \quad \alpha_{24}^{ST} = \alpha_{24}^{ST}, \quad \alpha_{34}^{ST} = \alpha_{34}^{ST}, \quad \theta_1^{ST} = \theta_1^{ST}, \\
\alpha_{12}^{ST} &= \alpha_{12}^{ST}, \quad \alpha_{13}^{ST} = \alpha_{13}^{ST}, \quad \alpha_{23}^{ST} = \alpha_{23}^{ST}, \quad \alpha_{24}^{ST} = \alpha_{24}^{ST}, \quad \alpha_{34}^{ST} = \alpha_{34}^{ST}, \quad \theta_1^{ST} = \theta_1^{ST}, \quad (14)
\end{align*}
\]

Take spatial triangle 123 for example, the geometry conditions on links 12 and 23, i.e. \( \alpha_{12}^{ST}, \alpha_{12}^{ST}, \alpha_{23}^{ST}, \alpha_{23}^{ST}, \theta_1^{ST} \) and \( R_2^{ST} \) are pre-defined design parameters. The rest of the parameters related to link 31 can be derived with Eq. (9).

Bennett linkage 1346 in asymmetric joint-axis setup, as the original Bennett linkage, is used as an intermediate bridge to connect these two spatial triangles. Bennett linkage 1346 and spatial triangle 123 share the common link 31 for merging. Note that the joint axes on the links to be merged should be kept along the same directions. This is the same for merging Bennett linkage 1346 and spatial triangle 456 on link 46. Therefore, the geometry conditions of the Bennett linkage 1346 in asymmetric
The geometric conditions are directly related to those of the spatial triangle and Bennett linkage as

\[
\alpha_{31}^{AB} = \alpha_{64}^{ST} = \alpha_{31}^{ST} = \alpha_{64}^{AB}, \quad \alpha_{34}^{AB} = \alpha_{61}^{ST} = \alpha_{34}^{AB},
\]

and

\[
\sin \frac{\alpha_{34}^{AB}}{a_{34}^{ST}} = \sin \frac{\alpha_{34}^{ST}}{a_{34}^{ST}}, \quad R_i^B = 0 (i = 1, 3, 4 \text{ and } 6).
\]

In Fig. 4, after removing the common links and joints marked in dash lines, the rest form a single-loop overconstrained 6R linkage. Its geometry conditions are directly related to those of the spatial triangle and Bennett linkage as

\[
a_{12} = a_{45} = a_{12}^{ST}, \quad a_{23} = a_{56} = a_{23}^{ST}, \quad a_{34} = a_{61} = a_{34}^{ST},
\]

\[
\alpha_{12} = \alpha_{45} = \alpha_{12}^{ST}, \quad \alpha_{23} = \alpha_{56} = \alpha_{23}^{ST}, \quad \alpha_{34} = \alpha_{61} = \alpha_{34}^{ST},
\]

\[
\sin \frac{\alpha_{34}}{\alpha_{34}^{ST}} = \sin \frac{\alpha_{34}^{ST}}{\alpha_{34}^{ST}},
\]

\[
R_1 = R_4 = R_1^{ST}, \quad R_2 = R_5 = R_2^{ST}, \quad R_3 = R_6 = R_3^{ST}.
\]

Thus, this 6R linkage belongs to the general line-symmetric Bricard linkage with \(\theta_{2,5}\) fixed at \(\theta_2^{ST}\).
The explicit closure equations of this 6R linkage could be derived by analysing the detailed relationship among $\theta_i$, $\theta_{ST}^i$ and $\theta_{AB}^i$. For example, the construct process of joint 3 in Fig. 5 determines

$$\theta_3 = \theta_{ST}^3 + \theta_{AB}^3 + \pi.$$ (17)

As a result, the compatibility conditions for the asymmetric 6R linkage are

$$\begin{align*}
\theta_1 &= \theta_{ST}^1 + \theta_{AB}^1 - \pi, \\
\theta_2 &= \theta_{ST}^2, \\
\theta_3 &= \theta_{ST}^3 + \theta_{AB}^3 + \pi, \\
\theta_4 &= \theta_{ST}^1 - \theta_{AB}^1 - \pi, \\
\theta_5 &= \theta_{ST}^2, \\
\theta_6 &= \theta_{ST}^3 - \theta_{AB}^3 - \pi.
\end{align*}$$ (18)

By substituting the closure equations of the spatial triangle in Eq. (2) and that of the asymmetric Bennett linkage in Eqs. (4a)–(4c) into Eq. (18), we can derive the explicit closure equations of this linkage as follows.

$$\begin{align*}
\theta_1 + \theta_4 &= 2\theta_{ST}^1, \\
\theta_2 &= \theta_{ST}^2, \\
\theta_3 + \theta_6 &= 2\theta_{ST}^3, \\
\theta_3 &= 2\tan^{-1}\left(\frac{\sin\alpha_{14} + \alpha_{31}^{ST}}{2\sin\frac{\alpha_{14} - \alpha_{31}^{ST}}{2}}\tan\frac{\theta_1 - \theta_{ST}^1}{2}\right) + \pi.
\end{align*}$$ (19)

where $\theta_{ST}^i$ and $\alpha_{31}^{ST}$ are determined in Eq. (2) with pre-defined design parameters $\alpha_{12,23}^{ST}$, $a_{12,23}^{ST}$ and $\theta_{ST}^2$. The kinematic paths are plotted in Fig. 6 using Eq. (19). Note that $\theta_2,5$ is constrained to a fixed design parameter of $\theta_{ST}^2$ during the full circle movement, which corresponds to the pre-defined configurations of joints 2 and 5 in the spatial triangles. Although the geometry conditions

![Fig. 5. The compatibility condition of joint 3 in the first reconfigurable 6R linkage.](image)

![Fig. 6. The kinematic paths of the asymmetric 6R linkage.](image)
of the 6R linkage are line-symmetric in Eqs. (16a)–(16d), the resultant linkage is still kinematically asymmetric with $\theta_1 \neq \theta_4$, $\theta_2 = \theta_5$ and $\theta_3 \neq \theta_6$ in Eq. (19), which is an inherited property from the Bennett linkage in asymmetric setup.

The linkage’s singular values are plotted in Fig. 7 using the Singular Value Decomposition method [41,42], in which the fifth singular value falls to zero at $B_{I}$ and $B_{II}$, indicating possible bifurcation behaviours. At point $B_{I}$, it is found that the linkage could bifurcate into an operation form with six active revolute joints, with line-symmetric property, see Fig. 8. The line of symmetry is shown as the central line in front view and the dashed dot in top view.

By using the SVD method, this linkage’s kinematic paths are shown in Fig. 9 and the singular values in Fig. 10. It is found that the kinematic paths are in correspondence to the closure equations in Eq. (8), which indicates that this linkage is a Form I general line-symmetric Bricard linkage. To differentiate the linkages, we name the linkage in Fig. 4 as the Bennett linkage form.

![Fig. 7. The SVD results of the asymmetric 6R linkage.](image)

![Fig. 8. The configuration of the Form I of the asymmetric 6R linkage when $\theta_1 = 14.9739\pi / 180$ and $\theta_2 = (\theta_2^{ST} =) - 70.0000\pi / 180$.](image)

![Fig. 9. The kinematic paths of the Form I of the asymmetric 6R linkage. Note that the figures are plotted in the region that $\theta_1 \in [\pi / 36, 5\pi / 36]$.](image)
linkage in Fig. 8 as the Form I linkage. In Fig. 10, the fifth singular value falls to zero at $B_I$, which is in accordance to the SVD results in Fig. 7. The location of $\theta_{BI}$ can be determined analytically in Eq. (20) by substituting the pre-defined revolute parameter $\theta_{ST}^2$ on joint 2 in the Bennett linkage form into the closure equations of the Form I general line-symmetric Bricard linkage in Eq. (8). The solution to Eq. (20) is derived in Appendix A.

$$\tan \frac{\theta_{ST}^2}{2} = \frac{-B_{term_2} + \sqrt{B_{term_2}^2 - 4A_{term_2} \cdot C_{term_2}}}{2A_{term_2}}$$

Fig. 10. The SVD results of the Form I of the asymmetric 6R linkage.

Fig. 11. The configuration of the Form II of the asymmetric 6R linkage when $\theta_1 = 176.7291\pi / 180$ and $\theta_2 = (\theta_{ST}^2) = -70.0000\pi / 180$.

Fig. 12. The kinematic paths of the Form II of the asymmetric 6R linkage.
Fig. 13. The SVD results of the Form II of the asymmetric 6R linkage.

Fig. 14. The transitions of the asymmetric 6R linkage with multiple operation forms. (a)–(c) are the motion sequence of the Form I linkage; (d) is the bifurcation configuration between Form I linkage and the Bennett linkage form; (e)–(g) are the motion sequence of the Bennett linkage form; (h) is the bifurcation configuration between the Bennett linkage form and the Form II linkage; (i)–(k) are the motion sequence of the Form II linkage.
On the other hand, the Bennett linkage form could bifurcate at BII in Fig. 7 into another operation form with six active revolute joints, see Fig. 11, whose kinematic paths are plotted in Fig. 12 and SVD results in Fig. 13. This linkage’s kinematic paths are in correspondence to the closure equations of the Form II general line-symmetric Bricard linkage in Eq. (9) and therefore named as the Form II linkage. From the SVD results in Fig. 13, the fifth singular value falls to zero at BII, which is in accordance to the SVD results in Fig. 7. The position of $\theta^{ST}_{BII}$ can be analytically determined in Eq. (21) by substituting the pre-defined revolute parameter $\theta^2_{ST}$ on joint 2 in the Bennett linkage form into the closure equations of the Form II general line-symmetric Bricard linkage in Eq. (9), whose solution is also included in the Appendix A.

$$\tan \frac{\theta^{ST}_{BII}}{2} = \frac{-B_{term2} - \sqrt{B_{term2}^2 - 4A_{term2} \cdot C_{term2}}}{2A_{term2}}$$ (21)

The full map of bifurcation for the constructed 6R linkage is plotted in Fig. 14. The Bennett linkage form can bifurcate into the Form I or Form II linkages on different bifurcation points, but the Form I and II linkages cannot bifurcate into each other directly. Therefore, we successfully introduce the operation form of a Bennett linkage with only 4 active revolute joints to bridge the two forms of the general line-symmetric Bricard linkage. The use of Bennett linkage in asymmetric joint-axis setup disrupts the line-symmetric relationship among the kinematic variables and enables the reconfiguration capability.

4. The line-symmetric 6R linkage

As illustrated in Fig. 15, two identical spatial triangles are the same as those in Fig. 3, and the intermediate bridge is a Bennett linkage in line-symmetric joint-axis setup, whose geometry conditions are

$$\begin{align*}
\alpha^{LB}_{31'} &= \alpha^{AB}_{31'}, \quad \alpha^{LB}_{34'} = \alpha^{AB}_{34'}, \\
\alpha^{LB}_{61'} &= \alpha^{AB}_{61'}, \quad \alpha^{LB}_{64'} = \alpha^{AB}_{64'}, \\
\sin \alpha^{LB}_{31'} &= -\sin \alpha^{LB}_{34'}, \quad R^{LB}_i = 0 (i = 1', 3', 4' \text{ and } 6').
\end{align*}$$ (22)
In Fig. 16, after removing the common links and joints in dash lines, the rest form the second 6R linkage. Its geometry conditions are

\[
\begin{align*}
    \alpha_{1'2'} &= \alpha_{3'5'} = \alpha_{4'6'}^{ST}, \\
    \alpha_{2'3'} &= \alpha_{5'6'}^{ST}, \\
    \alpha_{3'4'} &= \alpha_{6'1'}^{LB}, \\
    \sin \alpha_{3'4'} &= \sin \alpha_{3'5'}^{ST}, \\
    R_{1'} &= R_{4'} = R_{1}^{ST}, \\
    R_{2'} &= R_{2} = R_{2}^{ST}, \\
    R_{3'} &= R_{3}^{ST}.
\end{align*}
\]

which also belong to the general line-symmetric Bricard linkage with \( \theta_{2', 5'} \) fixed at \( \theta_{2'}^{ST} \). Following the same procedure as the foregoing section, the closure equations of this 6R linkage are derived as follows,

\[
\begin{align*}
    \theta_1 &= \theta_{4'}, \\
    \theta_2 &= \theta_{2'}, \\
    \theta_3 &= \theta_{6'} = \theta_3^{ST} - 2 \tan^{-1} \left( \frac{\cos \frac{\alpha_{3'4'} + \alpha_{3'5'}^{ST}}{2} \tan \frac{\theta_1 - \theta_{1'}^{ST}}{2}}{\cos \frac{\alpha_{3'4'} - \alpha_{3'5'}^{ST}}{2}} \right) + \pi.
\end{align*}
\]

where \( \theta_{1', 2'} \) and \( \alpha_{3'5'}^{ST} \) are determined in Eq. (2) with pre-defined design parameters \( \alpha_{12', 23', 24}^{ST} \) and \( \theta_{2'}^{ST} \). Both the geometric conditions and the kinematic variables of the second 6R linkage are in line symmetry. Its kinematic paths are plotted in Fig. 17 using Eq. (24), in which \( \theta_{2', 5'} \) are constrained to the design parameter of \( \theta_{2'}^{ST} \) during the full circle movement. Further investigation shows that the kinematic paths in Fig. 17 are in correspondence to the closure equations of the Form II general line-symmetric Bricard linkage in Eq. (9). Therefore, this Bennett form in Fig. 16 is actually the Form II of the general line-symmetric Bricard linkage with two fixed joints. Its singular values are plotted using SVD method in Fig. 18. There is no singular configuration, which is different from the 6R linkage in asymmetric setup. The use of Bennett linkage in line-symmetric setup preserved the line-symmetric relationship among the kinematic variables. Thus, the resultant linkage shares the same kinematic property as the general line-symmetric Bricard linkage, which is not reconfigurable [40]. The kinematic paths of the corresponding Form I linkage is shown in Fig. 19 using Eq. (8), which confirms that there is no common configuration between the Forms I and II linkages of the second 6R linkage we just constructed.

5. Conclusion

In this paper, the feasibility of reconfiguration between the general line-symmetric Bricard linkage and the Bennett linkage is explored based on their symmetry property. By using spatial triangles and Bennett linkages in different symmetry joint-axis setups as the building blocks, two 6R linkages have been constructed.
The first 6R linkage is achieved by connecting two identical spatial triangles with a Bennett linkage in asymmetric joint-axis setup, and then removing the redundant links inside to form a closed-loop overconstrained spatial linkage. The kinematic singularity analysis reveals that this linkage is actually in a Bennett linkage form and reconfigurable to the Forms I and II line-symmetric Bricard linkage through bifurcation points. Therefore, it is a reconfigurable linkage.

By replacing the Bennett linkage in asymmetric joint-axis setup with a Bennett linkage in line-symmetric setup in the construction, the second 6R linkage has been formed. The second linkage shares the same kinematic properties with the general line-symmetric Bricard linkage and it is not reconfigurable between the two linkage forms. The constructed 6R linkage is a special case of the Form II line-symmetric Bricard linkage with two fixed joints. A summary of these two linkages is listed in Table 1.

Here, the construct method for the proposed reconfigurable Bricard linkage has been developed. For a given line-symmetric Bricard linkage, either reverse-construct method or SVD method can be applied to testify whether it could be reconfigured into a
Bennett linkage. This work not only explores the design method for a reconfigurable mechanism between 4R and 6R overconstrained linkages, but also sets up a connection between the Bennett linkage and the line-symmetric Bricard linkage.

Acknowledgement

C.-Y. Song would like to thank NTU for providing the University Graduate Scholarship during his PhD study. This work is financially supported by the Natural Science Foundation of China (Projects No. 51275334 and No. 51290293).

Appendix A. The solutions to Eqs. (20) and (21)

The solutions to Eqs. (20) and (21) are derived from the following equations:

$$\tan \frac{\theta_{ST}^2}{2} = \frac{-B_{term} \pm \sqrt{B_{term}^2 - 4A_{term} \cdot C_{term}}}{2A_{term}}.$$  \((25)\)

The above equation could be rewritten into the following form,

$$A_{term} \cdot \tan^2 \frac{\theta_{ST}^2}{2} + B_{term} \cdot \tan \frac{\theta_{ST}^2}{2} + C_{term} = 0,$$  \((26)\)

where \(A_{term}, B_{term}\) and \(C_{term}\) are functions of \(\theta_1\) in Eqs. (10) and (12). By substituting Eqs. (10) and (12) into Eq. (26), we will get the following equation about \(\theta_{BI}^1\):

$$A \cdot \sin \theta_{BI}^1 + B \cdot \cos \theta_{BI}^1 + C = 0,$$  \((27)\)

where

$$\begin{cases}
A = (A_2 - F_2) \tan \frac{\theta_{ST}^2}{2} + 2E_2 \tan \frac{\theta_{ST}^2}{2} + (A_2 - F_2), \\
B = (B_2 - H_2) \tan \frac{\theta_{ST}^2}{2} + 2G_2 \tan \frac{\theta_{ST}^2}{2} + (B_2 - H_2), \\
C = (L_2 - D_2) \tan \frac{\theta_{ST}^2}{2} + 2C_2 \tan \frac{\theta_{ST}^2}{2} + (L_2 - D_2). 
\end{cases}$$  \((28)\)

After half-tangent transformation of the \(\sin \theta_{BI}^1\) and \(\cos \theta_{BI}^1\) in Eq. (28), we can derive that

$$(C - B) \cdot \tan^2 \frac{\theta_{BI}^1}{2} + 2A \cdot \tan \frac{\theta_{BI}^1}{2} + (C + B) = 0.$$  \((29)\)

By solving Eq. (29), we can derive that

$$\tan \frac{\theta_{BI}^1}{2} = \frac{-A \pm \sqrt{A^2 + B^2 - C^2}}{C - B}.$$  \((30)\)

Therefore, the positive result of Eq. (30) is the solution to Eq. (20), while the negative result of Eq. (30) is the solution to Eq. (21). The symbols are determined in Eqs. (12) and (28).
References